

NEHRU COLLEGE OF ENGINEERING AND RESEARCH CENTRE

(Accredited by NAAC, Approved by AICTE New Delhi, Affiliated to APJKTU)

Pampady, Thiruvilwamala(PO), Thrissur(DT), Kerala 680 588

DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING



COURSE MATERIALS

MAT 203 DISCRETE MATHEMATICS

VISION OF THE INSTITUTION

To mould our youngsters into Millennium Leaders not only in Technological and Scientific Fields but also to nurture and strengthen the innate goodness and human nature in them, to equip them to face the future challenges in technological break troughs and information explosions and deliver the bounties of frontier knowledge for the benefit of humankind in general and the down-trodden and underprivileged in particular as envisaged by our great Prime Minister Pandit Jawaharlal Nehru

MISSION OF THE INSTITUTION

To build a strong Centre of Excellence in Learning and Research in Engineering and Frontier Technology, to facilitate students to learn and imbibe discipline, culture and spirituality, besides encouraging them to assimilate the latest technological knowhow and to render a helping hand to the under privileged, thereby acquiring happiness and imparting the same to others without any reservation whatsoever and to facilitate the College to emerge into a magnificent and mighty launching pad to turn out technological giants, dedicated research scientists and intellectual leaders of the society who could prepare the country for a quantum jump in all fields of Science and Technology

ABOUT DEPARTMENT

- ◆ Established in: 2002
- ◆ Course offered: B.Tech COMPUTER SCIENCE AND ENGINEERING
: M.TECH COMPUTER SCIENCE AND ENGINEERING
:M.TECH CYBER SECURITY
- ◆ Approved by AICTE New Delhi and Accredited by NAAC
- ◆ Affiliated to the University of A P J Abdul Kalam Technological University.

DEPARTMENT VISION

Producing Highly Competent, Innovative and Ethical Computer Science and Engineering Professionals to facilitate continuous technological advancement

DEPARTMENT MISSION

- M1:** To Impart Quality Education by creative Teaching Learning Process
- M2:** To Promote cutting-edge Research and Development Process to solve real world problems with emerging technologies.
- M3:** To Inculcate Entrepreneurship Skills among Students
- M4:** To cultivate Moral and Ethical Values in their Profession

PROGRAMME EDUCATIONAL OBJECTIVES

- PEO1:** Graduates will be able to Work and Contribute in the domains of Computer Science and Engineering through lifelong learning.
- PEO2:** Graduates will be able to Analyse, design and development of novel Software Packages, Web Services, System Tools and Components as per needs and specifications.
- PEO3:** Graduates will be able to demonstrate their ability to adapt to a rapidly changing environment by learning and applying new technologies.
- PEO4:** Graduates will be able to adopt ethical attitudes, exhibit effective communication skills, Teamwork and leadership qualities.

COURSE OUTCOMES	
C203.1	Learn the fundamentals of enumeration or counting techniques and methods of arrangement and derangements
C203.2	Learn the fundamentals of propositional logic and predicate calculus and apply to test the validity of the statements
C203.3	Learn the ideas of the relation,,function equivalence relation and posets ,and its applications
C203.4	Understand recurrence relation and apply the method of solving different types of recurrence relations using generating functions
C203.5	Understand Fundamentals of Algebraic structures its properties such as monoids and groups

PROGRAM OUTCOMES (PO'S)

After the successful completion of the Course, B.Tech. Computer Science and Engineering, **Graduates can able to**

PO1: Engineering Knowledge: Apply the knowledge of Mathematics, Science, to solve complex engineering problems related to Design, Development, Testing and Maintenance of Software and System Tools

PO2: Problem Analysis: Identify, Analyse and Formulate complex problems to achieve significant conclusions by applying Mathematics, Natural Sciences and Computer Science and Engineering Principles and Technologies.

PO3: Design/Development of solutions: Design and construct software system, programme, component or process to meet the desired needs within the realistic constraints.

PO4: Conduct investigations of complex problems: Use research based knowledge and research methods to perform Literature Survey, design experiments for complex problems in designing, developing and maintaining computing systems, collect data from experimental outcome, analyse and interpret the interesting patterns and to provide effective conclusions.

PO5: Modern tool usage: Create, select and apply appropriate state-of-the-art Tools and Techniques in designing, developing, testing and validating Computing Systems, Tools and Components.

PO6: The engineer and society: Assess the societal, health, security, legal and cultural issues that might arise during Professional Practice in Computer Science and Engineering.

PO7: Environment and sustainability: Demonstrate the knowledge of sustainable development of Software, Components, Tools, Computing Systems and Solutions with an understanding of the impact of these engineering solutions on society and environment.

PO8: Ethics: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice of Computer Science and Engineering.

PO9: Individual and Team Work: Function effectively as an individual, and as a member or leader in multi-disciplinary teams, and strive to achieve common goals.

PO10:Communication: Communicate effectively with engineering community and society and be able to comprehend and write effective reports and documents, make effective presentations and give and receive clear instructions.

PO11:Project Management and Finance: Apply knowledge of the Engineering and Management principles to one's own work, as a member and leader in a team, to manage projects in Multidisciplinary Teams.

PO12:Life-long learning: Recognize the need for lifelong learning to cope up with the rapidly emerging Cutting Edge Technologies in Computer Science and Engineering and its allied Engineering application domains.

CO'S	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
C203.1	3	2	3	2	1	-	-	-	-	-	-	1
C203.2	3	2	3	2	3	-	-	-	-	-	-	1
C203.3	3	2	2	-	2	-	-	-	-	-	-	1
C203.4	3	2	3	1	-	-	-	-	-	-	-	-
C203.5	3	2	2	1	1	-	-	-	-	-	-	-

HIGH	3
MODERATE	2
LOW	1
NIL	-

PROGRAM SPECIFIC OUTCOMES (PSO'S)

1). PSO1: Analysis Skills: Ability to Formulate and Simulate Innovative Ideas to provide software solutions for Real-time Problems.

2). PSO2: Design Skills : Ability to Analyse and design various methodologies for facilitating development of high quality System Software Tools and Efficient Web Design Models with a focus on performance optimization.

3). PSO3: Product Development : Ability to Apply Knowledge for developing Codes and integrating hardware/software products in the domains of Big Data Analytics, Web Applications and Mobile Apps

CO'S	PSO1	PSO2	PSO3
C203.1	2	2	
C203.2	3	3	
C203.3	2	3	
C203.4	2		
C203.5	2	2	

MATHEMATICS -3 rd Semester BTech
For Computer Science and Engineering and Information Technology
DISCRETE MATHEMATICS

MAT 203	COURSE NAME: DISCRETE MATHEMATICS	CATEGORY	L	T	P	CREDIT
		BASIC SCIENCE COURSE	3	1	0	4

Preamble:

This course introduces the concept of mathematical structures that are fundamentally discrete. The course enable the students to understand and apply the fundamentals of enumeration and counting techniques and different way of arrangements. The course introduce the concept of relations and functions. Propositional logic and predicate calculus are introduced so that the students can test the validity of statements. Methodsof applying recurrence relations to solve problems in different domains are introduced. An introduction to algebraic structures such as monoid and group.

Prerequisite: A soundbackground in higher secondary school Mathematics

Course Outcomes: After the completion of the course the student will be able to

CO 1	Learn the fundamentals of enumeration or counting techniques and methods of arrangements and derangements.
CO 2	Learn the fundamentals of propositional logic and predicate calculus and apply to test the validity of statements
CO 3	Learn the ideas of relations, functions equivalence relation and posets and it's applications
CO 4	Understand recurrence relation and apply the method of solving different type of recurrence relations using generating functions
CO 5	Understand Fundamentals of Algebraic structures its properties such as monoids and groups

Mapping of course outcomes with program outcomes

PO's	Broad area
PO 1	Engineering Knowledge
PO 2	Problem Analysis
PO 3	Design/Development of solutions
PO 4	Conduct investigations of complex problems
PO 5	Modern tool usage
PO 6	The Engineer and Society
PO 7	Environment and Sustainability
PO 8	Ethics
PO 9	Individual and team work
PO 10	Communication
PO 11	Project Management and Finance
PO 12	Lifelong learning

	PO 1	PO 2	PO 3	PO 4	PO 5	PO 6	PO 7	PO 8	PO 9	PO 10	PO 11	PO 12
CO 1	3	2	3	2	1					2		1
CO 2	3	2	3	2	3					2		2
CO 3	3	2	2		2					2		2
CO 4	3	2	3	1						2		
CO 5	3	2	2	1	1					2		1

Assessment Pattern

Bloom's Category	Continuous Assessment Tests(%)		End Semester Examination(%)
	1	2	
Remember	10	10	10
Understand	30	30	30
Apply	30	30	30
Analyse	20	20	20
Evaluate	10	10	10
Create			

End Semester Examination Pattern: There will be two parts; Part A and Part B. Part A contain 10 questions with 2 questions from each module, having 3 marks for each question. Students should answer all questions. Part B contains 2 full questions from each module of which student should answer any one. Each question can have maximum 2 sub-divisions and carry 14 marks.

Course Level Assessment Questions

Course Outcome 1 (CO1):

- 1) How many possible arrangements are there for the letters in MASSASAUGA in which 4 As are together?
- 2) Determine the coefficient of xyz^2 in $(x + y + z)^4$ using combinatorial arguments
- 3) Find the number of integers between 1 and 10000 inclusive which are divisible by none of 5, 6 or 8
- 4) List all the derangements of 1,2,3,4,5 where the first three numbers are 1,2, 3 in some order

Course Outcome 2 (CO2):

- 1) Determine the truth value of the implication 'if $3 + 4 = 12$ then $3 + 2 = 6$ '
- 2) Determine the truth value of $\exists x \forall [xy = 2]$ if the universe comprises all non-zero integers.
- 3) Use truth table to verify $p \rightarrow (q \wedge r) \Leftrightarrow (p \rightarrow q) \wedge (p \rightarrow r)$
- 4) Let (x) and (x) be open statements in the variable x with a given universe. Prove that $[\forall x p(x) \vee \forall x q(x)] \Rightarrow \forall x [p(x) \vee q(x)]$

Course Outcome 3 (CO3):

1. Give an example of a function $f; A \rightarrow B$ and $A_1, A_2 \subseteq A$ for which $f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$
2. If $A = \{1, 2, 3, 4\}$, give an example of a relation R that is reflexive and symmetric but not transitive
3. Define the relation R on the set $\mathbf{Z}$ by aRb if $a - b$ is a non-negative even integer. Verify that
 - i. R defines a partial order on $\mathbf{Z}$. Is this partial order a total order?
4. Draw the Hasse diagram of the poset $(\wp(\mathcal{U}), \subseteq)$ where $\mathcal{U} = \{1,2,3,4\}$

Course Outcome 4 (CO4):

1. Find the unique solution for the recurrence relation $4a_n - 5a_{n-1} = 0, n \geq 1$
2. The number of bacteria in a culture is 1000 (approximately) and the number increases 250% every 2 hours. Use a recurrence relation to determine the number of bacteria after one day.
3. Solve the recurrence relation $2a_n = 7a_{n-1} - 3a_{n-2}, a_0 = 2, a_1 = 5$.
4. Solve $a_{n+1} - a_n = 3n^2 - n, n \geq 3, a_0 = 3$.

Course Outcome 5 (CO5):

1. State two monoids for $P(S)$, the power set of S
2. Give an example for a finite group.
3. Show that $(A, *)$ is an abelian group where $A = \{a \in \mathbf{Q} \mid a \neq -1\}$ and for any $a, b \in A, a * b = a + b + ab$
4. If G is a group with $(G) = 2p$ where p is prime, prove that every subgroup of G is cyclic.

Syllabus

Module 1: (Fundamentals of Counting Theory)

(Relevant topics from sections 1.1, 1.2, 1.3, 1.4, 5.5, 8.1, 8.2, 8.3)

The Rule of Sum – Extension of Sum Rule - The Rule of Product - Extension of Product Rule - Permutations

,Combinations, The Binomial Theorem (without proof),Combination with Repetition. The Pigeonhole Principle, The principle of Inclusion and Exclusion Theorem (Without Proof) Generalisation of the principle.Derangements-Nothing in it's right place. (9 hours)

Module- 2 (Fundamentals of Logic)

(Relevant topics from sections 2.1, 2.2, 2.3, 2.4.)

Mathematical logic:Basic connectives and truth table. Statements – Logical Connectives – Tautology – Contradiction.Logical Equivalence: The Laws of Logic, The principle of duality- Substitution Rules – The implication-The Contrapositive- the Converse – the Inverse.

Logical Implication - Rules of Inference, The use of Quantifiers: Open Statement- Quantifier-Logically Equivalent – Contrapositive –Converse –Inverse , Logical equivalences and implications for quantified statement- implications ,negation (9 hours)

Module 3 (Relations and Functions)

(Text 1: Relevant topics from sections 5.1, 5.2, 7.1,7.2,7.3,7.4, 7.6)

Cartesian Product - Binary Relation, Function – domain – range, One to One function, Image-restriction, Properties of Relations- Reachability Relations- Reflexive Relations-Symmetric Relations-Transitive relations - Antisymmetric Relations,Partial Order relations, Equivalence Relation-irreflexiverelations.Computer Recognition: Zero-one Matrices - Composite Relations- Zero One Matrix-Relation Matrix. Partially ordered Set –HasseDiagram-Maximal-Minimal Element-Least upper bound (lub)- Greatest Lower bound(glb) **(From section 7.2, graph theory excluded.From 7.3,Topological sorting Algorithm- excluded)**

Equivalence Relations andPartitions - Equivalence Class, Lattice- Dual Lattice – Sub lattice – Properties of glb and lub – Properties of Lattice - Special Lattice : Complete Lattice – Bounded Lattice – Completed Lattice – Distributive Lattice (9hours)

Module- 4 (Generating Functions and Recurrence Relations)

(Relevant topics from sections 9.1,9.2, 9.4, 10.1, 10.2, 10.3)

Generating Function - Definition and Examples - Calculation techniques, Exponential generating function. First order linear recurrence relations with constant Coefficients-homogeneous-nonhomogeneous- Solution. The second-order linear recurrence relation with Constant Coefficients - homogeneous- Nonhomogeneous – Solution.- Three cases, The Nonhomogeneous Recurrence Relation

-First	order-	Second	Order
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(9hours)

Module-5 (Algebraic Structures)

(Relevant topics from sections 15.1, 15.2, 15.3, 15.4, 15.5)

Algebraic system-Properties- Homomorphism and Isomorphism. Semigroup and monoid– cyclic monoid, sub semigroup and sub monoid- Homomorphism and Isomorphism of Semigroup and monoids. Group- Elementary properties- subgroup- symmetric group on three symbols-The direct product of two groups. Group Homomorphism- Isomorphism of groups, Cyclic group. Right coset- left cosets- Lagrange's Theorem(9 hours)

Text Book

1. Discrete and Combinatorial Mathematics (An Applied Introduction), Ralph P Grimaldi, B V Ramana , 5th Edition, Pearson

Reference Books

- 1) Kenneth H. Rosen, Discrete Mathematics and Its Applications with Combinatorics and Graph Theory, Seventh Edition, MGH, 2011
- 2) Tremblay J.P and Manohar R, “Discrete Mathematical Structures with Applications to Computer Science”, Tata McGrawHill Pub. Co. Ltd., New Delhi, 2003.
- 3) Bernard Kolman, Robert C. Busby, Sharan Cutler Ross, “Discrete Mathematical Structures”, Pearson Education Pvt Ltd., New Delhi, 2003
- 4) Kenneth H. Rosen, “Discrete Mathematics and its Applications”, 5/e, Tata Mc Graw Hill Pub. Co. Ltd, New Delhi 2003
- 5) Richard Johnsonbaugh, “Discrete Mathematics”, 5/e, Pearson Education Asia, New Delhi, 2002.
- 6) Joe L Mott, Abraham Kandel, Theodore P Baker, “Discrete Mathematics for Computer Scientists and Mathematicians”, 2/e, Prentice-Hall India, 2009.

Assignments:

Assignment should include specific problems highlighting the applications of the methods introduced in this course in science and engineering.

Course Contents and Lecture Schedule

No	Topic	No. of Lectures
1	Fundamentals of Counting Theory 9 hours	
1.1	The Pigeonhole Principle	1
1.2	The Rule of Sum – Extension of Sum Rule -	1
1.3	The Rule of Product - Extension of Product Rule - Permutations	1
1.4	Combinations, The Binomial Theorem (Without proof) Combination with repetition	2
1.5	The principle of Inclusion and Exclusion Theorem (Without Proof) Generalisation of the principle	2
1.6	Derangements	2
2	Fundamentals of Logic 9 hours	
2.1	Mathematical logic -Basic Connectives and Truth Table: Statements – Logical Connectives – Tautology - Contradiction	3
2.2	Logical Equivalence: The Laws of Logic- The principle of duality- Substitution Rules	2
2.3	The implication-The Contrapositive, the Converse – the Inverse	1
	Logical Implication : Rules of Inference – Logical Implication The Use of Quantifiers: Open Statement- Quantifier	1
2.4	Logically Equivalent – Contrapositive – The Converse – The Inverse – Logical Implications – Negation	2

3	Relations and Functions 9 hours	
3.1	Cartesian Product - Binary Relation Function – Domain – Range – One to One Function Image- Restriction	1
3.2	Reachability Relations, Reflexive Relations-Symmetric Relations- Transitive relations - Antisymmetric Relations -	1
3.3	Partial Order relations -Equivalence Relation- Irreflexive Relations.	1
3.4	Zero-one Matrices and directed graphs- Composite Relations- Zero One Matrix-Relation Matrix - Directed Graph - Strongly Connected Directed Graph. Partially ordered Set - Hasse Diagram- Maximal-Minimal Element-Least Upperbound- Greatest Lower Bound	2
3.5	Partially ordered Set - Hasse Diagram-Maximal-Minimal Element- Least Upperbound- Greatest Lower Bound	2
3.6	Equivalence Relation - Equivalence Class	1
	Lattice- Dual Lattice – sublattice – Properties of glb and lub – Properties of Lattice - Special Lattice : Complete Lattice – Bounded Lattice – Completed Lattice – Distributive Lattice	1
4	Generating Functions and Recurrence Relations9 hours	
4.1	Generating Function - Definition and Examples, Exponential Generating Function.	2
4.2	First Order Linear Recurrence Relations with Constant Coefficients Homogeneous- Nonhomogeneous- Solution	2
4.3	The Second-Order Linear Recurrence Relation with Constant Coefficients -Homogeneous Nonhomogeneous Solution	3
4.4	The Nonhomogeneous Recurrence Relation -First order- Second Order	2

5	Algebraic Structures 9 hours	
5.1	Algebraic System-Properties- Homomorphism and Isomorphism	1
5.2	Cyclicmonoid, subsemigroup and submonoid- homomorphism and Isomorphism of Semigroup and Monoids.	2
5.3	ElementaryProperties- Subgroup- symmetric group on three symbols-The direct Product of two Groups	2
5.4	Group Homomorphism- Isomorphism- Cyclic group	2
5.5	Right coset- Left Cosets- Lagrange's Theorem	2

MODEL QUESTION PAPER

APJ ABDUL KALAM TECHNOLOGICAL UNIVERSITY

THIRD SEMESTER B. TECH DEGREE EXAMINATION

DISCRETE MATHEMATICS

(For Computer Science and Engineering and Information Technology)

(2019-Scheme)

Time: 3 hrs.

Max Marks: 100

PART A

(Answer all questions. Each question carries 3 marks)

1. In how many ways can eight men and eight women be seated in a row if any person may sit next to any other?
2. In how many ways can we distribute eight identical white balls into four distinct containers so that no container is left empty?
3. Negate and simplify the statement $p \rightarrow (\neg q \wedge r)$
4. For the universe of all quadrilaterals in the plane, let $s(x)$ and $e(x)$ denote the open statements

$s(x)$: x is a square $e(x)$: x is equilateral

Determine the truth value of $\forall [s(x) \rightarrow e(x)]$

5. If $A = \{1, 2, 3, 4\}$, find a relation on A that is reflexive and symmetric but not transitive -
6. Show that the lattice $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{1, 2, 3\}$ and $X = \{A, B, C\}$. Draw the Hasse diagram for $(X, \subseteq)$. Is $(X, \subseteq)$ a lattice? Why?
7. Find the coefficient of x^5 in $(1 - 2x)^{-7}$ using generating function.
8. Find a recurrence relation with initial condition that uniquely determine the sequence 6, -18, 54, -162,
9. State two monoids for the power set of any set S .
10. Find all subgroups of Z_5 .

PART B

(Answer one full question from each module. Each full question carries 14 marks)

MODULE 1

- 11 a) How many distinct four digit integers can one make from the digits 1, 3, 3, 7, 7 and 8?
- b) How many derangements are there for 1, 2, 3, 4, 5?
- 12 a) Determine the number of integer solutions of $x_1 + x_2 + x_3 + x_4 = 32$ where $x_i \geq 0$ $1 \leq i \leq 4$.
- b) Determine the number of positive integers n where $1 \leq n \leq 100$ and n is not divisible by 2, 3, or 5.

MODULE II

- 13 a) For $A = \{1, 2, 3, 4\}$, let R and S be the relations on A defined by
- $$R = \{(1,2), (1,3), (2,4), (4,4)\} \text{ and } S = \{(1,1), (1,2), (1,3), (2,3), (2,4)\}$$
- Find $R \circ S$, $S \circ R$ and R^3
- b) Let $A = \mathbb{R}^2$. Define R by $(x_1, y_1) R (x_2, y_2)$ if and only if $x_1 = x_2$. Prove that R is an equivalence relation and find the equivalence classes.
- 14 a) Find the number of ways to totally order the partial order of all positive integer of 1701
- b) Show that the lattice $(\mathbb{Z}^+, \leq)$ is distributive

MODULE III

- 15 a) Let $p(x)$, $q(x)$ and $r(x)$ be open statements for a given universe. Prove that the argument $\forall x[p(x) \rightarrow q(x)]$ and $\forall x[q(x) \rightarrow r(x)]$ logically implies $\forall x[p(x) \rightarrow r(x)]$. Give reason for each step
- 16 a) Prove that the premises $(\neg p \vee q) \rightarrow r$, $r \rightarrow (s \vee t)$, $\neg s \wedge \neg u$, $\neg u \rightarrow \neg t$ logically implies p .
- b) Let $p(x)$, $q(x)$ and $r(x)$ denote the following open statements
- $$p(x): x^2 - 8x + 15 = 0, \quad q(x): x \text{ is odd}, \quad r(x): x > 0$$
- For the universe of all integers, determine the truth or falsity of $\forall x[\neg p(x) \rightarrow \neg q(x)]$

MODULE IV

17 a) Determine the sequence generated by $(x) = 6e^{5x} - 3e^{2x}$

b) Use generating functions to determine how many four element subsets of

$S = \{1, 2, 3, \dots, 15\}$ contains no consecutive integers.

18 a) Solve the recurrence relation $a_n = a_{n-1} + a_{n-2}$, $n \geq 2$, $a_0 = 1$, $a_1 = 2$

b) Solve $a_{n+2} + 3a_{n+1} + 2a_n = 3$, $n \geq 0$, $a_0 = 0$, $a_1 = 1$.

MODULE V

19 a) Find the cyclic sub semigroup of $(M_2(\mathbb{Z}), \cdot)$ generated by the element $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

b) Find all elements of order 10 in $(\mathbb{Z}_{40}, +)$.

20 a) State and prove Lagrange's theorem for finite group.

b) For $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$, find the subgroup $k = \langle \beta \rangle$ generated by β and determine the

left cosets of $G = S_4$.

QUESTION BANK

MODULE 1				
S. No	Questions	CO	KL	PAGE NO:
1	(a) In how many ways can the letters in VISITING be arranged? (b) For the arrangements of part(a), how many have all three I's together?	CO1	K3	23
2	.Find the no. of ways in which 5 people A,B,C,D,E can be seated at a round table such that (a) A and B must always sit together (b) C and D must not sit together	CO1	K6	40
3	In how many ways can 3 men and 3 women be seated at a round table such that no two men are seated together?	CO1	K3	41
4	How many arrangements of the letters in TALLAHASSEE have no adjacent A's?	CO1	K3	26
5	A student is to answer 7 out of 10 questions on an examination. In how many ways can he make his selection if (a) there are no restrictions (b) he must answer the first two questions (c) he must answer atleast four of the first 6 questions	CO1	K3,K4	32
6	A committee of 12 is to be selected from 10 men and 10 women. In how many ways can the selection be carried out if (a) there are no restrictions (b) there must be 6 women and 6 men (c) there must be an even number of women (d) there must be atleast 8 men	CO1	K3,K4	33
7	Determine the coefficient of (a) xyz^2 in $(x + y + z)^4$ (b) xyz^2 in $(x + y + z + w)^4$	CO1	K6	42
8	Determine the coefficient of x^9y^3 in the expansion of (a) $(x + 2y)^{12}$ (b) $(2x - 3y)^{12}$	CO1	K6	45
9	In how many ways can we distribute 8 identical white balls into 4 distinct containers so that (a) no container is left empty? (b) the fourth container has an odd number of balls in it?	CO1	K3	46
10	State Pigeonhole principle. A school has 550 students. Show that at least two of them were born on the same day of the year.	CO1	K4	57

MODULE II				
S. No	Questions	CO	KL	PAGE NO:
1	Prove that $(P \wedge Q) \rightarrow (P \leftrightarrow Q)$ is a tautology.	CO2	K3	94
2	Show that $(a \vee b)$ follows logically from the premises $p \vee q, \quad p \vee q \rightarrow \neg r, \quad \neg r \rightarrow s \wedge \neg t, \text{ and}$ $s \wedge \neg t \rightarrow \quad \quad a \vee b$	CO2	K6	124
3	Represent the following sentence in predicate logic using quantifiers i) All men are mortal. ii) Every apple is red iii) Any integer is either positive or negative.	CO2	K2	132
4	Use the truth table to determine whether $p \rightarrow (q \wedge \neg q)$ and $\neg p$ are logically equivalent	CO2	K3	99
5	Construct a truth table for $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ and determine whether it is a tautology or not	CO2	K4	95
6	Negate and simplify $\exists x[(p(x) \vee q(x)) \rightarrow r(x)]$	CO2	K4	137
7	Construct an argument to show that the following premises imply the conclusion “it rained” “if it does not rain or if there is no traffic dislocation, then the sports day will be held and the cultural programme will go on.” “if the sports day is held, the trophy will be awarded” “the trophy was not awarded”	CO2	K6	126
8	Prove that the premises $\neg p \vee q \rightarrow r, r \rightarrow s \vee t, \neg s \wedge \neg u, \neg u \rightarrow \neg t$ logically implies p	CO2	K6	100
9	Show that $(\exists x)M(x)$ follows logically from the premises $(\forall x)H(x) \rightarrow M(x)$ and $(\exists x)H(x)$.	CO2	K3	146
10	Choose the obvious predicates and express in predicate logic. i) “Everybody loves somebody” ii) “All healthy people eat an apple a day”	CO2	K2	142

MODULE 1III				
S. No	Questions	CO	KL	PAGE NO:
1	Consider f, g, h are functions on integers such that $f(n) = n^2$, $g(n) = n + 1$, $h(n) = n - 1$. Determine (i) $f \circ g \circ h$ (ii) $g \circ f \circ h$ (iii) $h \circ f \circ g$	CO3	K3	199
2	Draw the Hasse diagram for the divisibility relation on the set $A = \{2, 3, 6, 12, 24, 36\}$.	CO3	K6	205
3	Determine whether the functions $f: \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(x) = 2x + 1$ is one to one and determine its range	CO3	K2	195
4	Let $f(x) = x+2$, $g(x) = x-2$ and $h(x) = 3x$ for x is in $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Find $g \circ f$, $f \circ g$, $(f \circ h) \circ g$, $h \circ g$.	CO3	K3	196
5	Define equivalence relation. Let R be a relation in the set of integers $\mathbb{Z}$ defined by $R = \{(x, y): x \in \mathbb{Z}, y \in \mathbb{Z}, (x - y) \text{ is divisible by } 6\}$. Prove that R is an equivalence relation	CO3	K4	171
6	Let $A = \{1, 2, 3, 4, \dots, 11, 12\}$ and let R be the equivalence relation on $A \times A$ defined by $(a, b) R (c, d)$ iff $a + d = b + c$. Prove that R is an equivalence relation and find the equivalence class of $(2, 5)$	CO3	K4	174
7	What is a chain Lattice? Explain Also Show that every chain is distributive lattice	CO3	K6	207
8	Let R and S be two relations on a set A . If R and S are symmetric prove that $R \cap S$ is also symmetric	CO3	K6	170
9	what is a chain lattice? Show that every chain is a distributive lattice	CO3	K3	210
10	Define a complimented lattice. Show that D_{42} with $'/'$ as order is a complimented lattice	CO3	K2	226

MODULE 1V				
S. No	Questions	CO	KL	PAGE NO:
1	Solve the recurrence relation $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 - 2r + 1$	CO4	K3	251
2	Provide one example of linear homogeneous recurrence relation. Mention the degree also	CO4	K6	230
3	Solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n$	CO4	K2	249
4	Solve the recurrence relation $T(k) - 7T(k-1) + 10T(k-2) = 6 + 8k$ with $T(0)=1$ and $T(1)=2$	CO4	K3	235
5	Solve the recurrence relation $a_r - 7a_{r-1} + 10a_{r-2} = 0$ for $r \geq 2$ given $a_0=0, a_1=$ using generating function	CO4	K4	231
6	Solve the recurrence relation $a_n - 3a_{n-1} + 2 = 0, a_0 = 1, n \geq 1$ using generating function	CO4	K4	232
7	Solve the recurrence relation $a_n = 2a_{n-1} + 2^n$ with $a_0 = 2$	CO4	K6	237
8	Solve the recurrence relation $a_n - 4a_{n-1} + 4a_{n-2} = (n+1)^2$ using generating function	CO4	K6	253
9	Solve the recurrence relation $a_{n+2} - 6a_{n+1} + 9a_n = 3(2^n) + 7(3^n)$	CO4	K3	237
10	Solve $f(n) = f(n-1) + f(n-2)$; $f(0)=1, f(1)=1$	CO4	K2	258

MODULE V				
S. No	Questions	CO	KL	PAGE NO:
1	What is a Monoid? SemiGroup? Explain with examples	CO5	K3	266
2	Let $(A,*)$ be a group.Show that $(ab)^{-1} = b^{-1}a^{-1}$	CO5	K6	268
3	Prove that the set Q of rational numbers other than 1 forms an abelian group with respect to the operation * defined by $a * b = a + b - ab$	CO5	K2	267
4	Show that subgroup of a cyclic group is cyclic.	CO5	K3	274
5	Let $(A,*)$ be a Group.Show that $(A,*)$ is an abelian group if and only if $a^2 * b^2 = (a * b)^2$	CO5	K4	270
6	Check whether the algebraic structure $(\mathbb{Z}_5, +5)$ defined over the set of positive integers is a semi group or not.	CO5	K4	267
7	Define Cosets and Lagranges theorem	CO5	K6	278
8	Show that the set $\{1,2,3,4,5\}$ is not a group under addition modulo 6	CO5	K6	269
9	Show that the set Q^+ of rational numbers forms an abelian group under the operation * defined by $a*b=\frac{1}{2}ab$, $a,b \in Q^+$	CO5	K3	267
10	If $\{R^+,X\}$ and $\{R,+\}$ are two semigroups in the usual notation,prove that the mapping $g(a):R^+ \rightarrow R$ defined by $g(a)=\log_e a$ is a semigroup isomorphism	CO5	K2	288

Permutations

Permutation is an arrangement of a no. of objects in some definite order taken some or all at a time. The total no. of permutation of n distinct objects taken ' r ' at a time is denoted by $n P_r$.

$$* \quad n P_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)\dots(n-r+1)$$

* No. of permutations of ' n ' things taken all at a time is $n P_n = n!$

* If there are ' n ' objects with n_1 of a first type, n_2 of a second type ... and n_r of an r^{th} type where $n_1 + n_2 + \dots + n_r = n$ then there are
$$\frac{n!}{n_1! n_2! \dots n_r!} \quad (\text{linear})$$
 arrangements of the given ' n ' objects.

Example:-

1. How many variable names of 8 letters can be formed from the letters a, b, c, d, e, f, g, h, i if no letter is repeated

There are 9 letters and 8 are to be selected.

∴ Total no. of variable names of 8 letters

$$\text{is } {}^9P_8 = \frac{9!}{(9-8)!} = \underline{\underline{9!}}$$

2. There are 10 persons called on an interview. Each one is capable to be selected for the job. How many permutations are there to select 4 from 10

There are 10 persons & 4 are to be selected

∴ Total no. of permutations to select 4

$$\text{persons is } = {}^{10}P_4 \Rightarrow \frac{10!}{(10-4)!} = \underline{\underline{5040}}$$

3. How many 6 digit numbers can be formed by using the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 if every number is to start with '30' with no digit repeated?

Ans:-

All the numbers begin with '30'. So we have to choose 4 digits from the remaining 7 digits.

$\therefore$ Total no. of numbers that begins with

$$30 \text{ is } {}^7P_4 = \frac{7!}{(7-4)!} = \frac{7!}{3!} = \underline{\underline{840}}$$

4. How many permutations can be made out of the word "COMPUTER"?

How many of these (i) begin with C
(ii) end with R (iii) begin with 'C' and end with 'R' (iv) C and R occupy the end places.

Ans: There are 8 letters in the word "COMPUTER" and all are distinct.

$\therefore$ Total no. of permutations of these letters is $8! = \underline{\underline{40320}}$

(i) permutation that begin with C

The 1st position can be filled in only one way and the remaining 7 letters can be arranged in 7! ways.

$\therefore$ Total no. of permutation starting with C are $= 1 \times 7! = \underline{\underline{5040}}$

(ii) permutation end with R.

The last position can be filled in only one way and the remaining 7 letters can be arranged in $7!$ ways.

$\therefore$ Total no. of permutations ~~ending~~ ^{ending} with R are $7! \times 1 = \underline{\underline{5040}}$

(iii) Permutation begin with C and end with

^R.
The 1st position C and the last position R can be filled by only in one way and the remaining 6 letters can be arranged in $6!$ ways.

$\therefore$ Total no. of permutation begin with C and end with R is $= 1 \times 6! \times 1 = \underline{\underline{720}}$

(iv) permutation in which C & R occupy the end places.

C & R occupy end places in $2!$ ways. & the remaining 6 letters in $6!$ ways.

$\therefore$ Total no. of permutation in which C & R occupy end places $= 2! \times 6! = 1440 //$

25

⑧ Determine the no. of permutations that can be made out of the letters of the word "PROGRAMMING"

There are 11 letters in the word 'PROGRAMMING' out of which G's, M's and R's are repeating twice each.

$$\therefore \text{Total no. of permutation is} = \frac{11!}{2! 2! 2!} = \underline{\underline{4989600}}$$

⑨ "MASSASAUGA" Determine the no. of permutations that can be made out of the letters of the above word.

There are 10 letters, in which

$$A's \rightarrow 4; S's \rightarrow 3; M \rightarrow 1; U \rightarrow 1; G \rightarrow 1$$

$$\therefore \text{no. of permutation is} = \frac{10!}{4! 3!} = \underline{\underline{25200}}$$

⑩ Determine the no. of permutations that can be made out of the letters of "MASSASAUGA" in which all four A's are together?

To get this result, we consider all arrangements of the seven symbols AAAA (one symbol), S, S, S, M, U, G

Consider AAAA as one, and remaining 6 digits. So there are a total of 7 letters in which S's are 3 $M \rightarrow 1$ $U \rightarrow 1$ & $G \rightarrow 1$

$\therefore$ No of permutation in which all

$$\text{four A's are together} = \frac{7!}{3! 1! 1!} = \underline{\underline{840}}$$

H.W
⑧ No. of permutations that can be made out of the letters of the word "PEPPER"

Ans: Total 6 letters.

$P \rightarrow 3, E \rightarrow 2, R \rightarrow 1$

$$\therefore \text{Total no. of permutation} = \frac{6!}{3! 2!} = \underline{\underline{60}}$$

⑨ How many arrangements are there of all the letters in "SOCIOLOGICAL"

⑩ In how many of the arrangements in part (a) are A and G adjacent

⑪ In how many of the arrangements in part (a) are all the ~~vowels~~ vowels adjacent.

Ans: ② Total 12 letters in which
 $O's \rightarrow 3, C's \rightarrow 2, I's \rightarrow 2, L's \rightarrow 2$
 $\therefore$ Total no. of permutations is $\frac{12!}{3! 2! 2! 2!}$

③ A & U should be adjacent that can be in $2!$ ways. Now take 'AU' as one symbol and the remaining letters S, O, C, I, O, L, O, C, L together we have 11 letters.

Out of these 11, $O \rightarrow 3, C \rightarrow 2, I \rightarrow 2, L \rightarrow 2$

$\therefore$ Total no. of permutations is,

$$2! \times \frac{11!}{3! 2! 2! 2!}$$

④ SOCIOLOGICAL

Vowels are O, I, O, O, I, A. These can be arranged in $\frac{6!}{3! 2!}$ ways since $O's \rightarrow 3, I's \rightarrow 2$

Now consider these vowels as one symbol and there remains S, C, L, U, C, L letters. So total 7 letters are there in which $C's \rightarrow 2, L's \rightarrow 2$

∴ No. of permutations here is, $\frac{7!}{2! 2!}$ 28

Thus no. of permutations in which all the vowels adjacent is, equal to,

$$\frac{6!}{3! 2!} \times \frac{7!}{2! 2!}$$

Permutation with repeated objects.

The no. of different permutations of n distinct objects taken ' r ' at a time when every object is allowed to repeat any no. of times is given by $\frac{n^r}{1}$.

Ex: How many 4 digit numbers can be formed by using the digits 2, 4, 6, 8 when repetition of digits is allowed.

We have 4 digits.

no. of ways filling unit's place = 4

no. of ways filling ten's place = 4

no. of ways filling hundred's place = 4

no. of ways filling thousand's place = 4

∴ total no. of 4 digit nos is $= 4 \times 4 \times 4 \times 4$
 $= 256 //$

29

Exa: (2) How many 2 digit even numbers can be formed by using the digits 1, 3, 4, 6, 8 when repetition of digits is allowed.

Ans: -

We have 3 even nos and 2 odd nos.

No. of ways filling unit's place = 3

No. of ways filling ten's place = 5

$\therefore$ Total no. of 2 digit even nos = $3 \times 5 = \underline{\underline{15}}$

Circular Permutation

The no. of circular permutation of n different object is $(n-1)!$

Exa: In how many ways 10 programmers can sit on a round table to discuss the project so that project leader and a particular programmer always sit together.

Ans: There are 10 programmers but project leader and a particular programmer always sit together. So both became single unit and hence there are total

9 units... These 9 units can be arranged on a round table in $8!$ ways. The 2 programmers can be arranged in $2!$ ways.

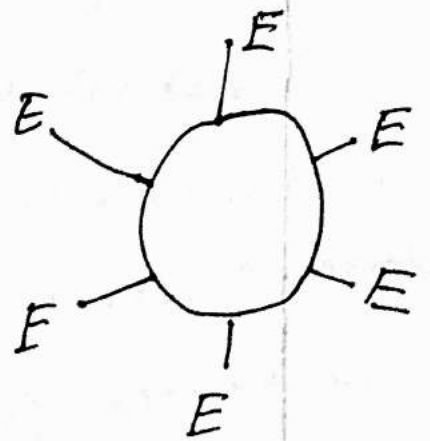
$\therefore$ Total no-of ways in which 10 programmers can sit on a round table is,

$$= 8! \times 2! \Rightarrow \underline{\underline{80640}}$$

Q. Determine the no-of ways in which 5 software Engineers and 6 electronic engineers can be seated at a round table so that no two software engineers can sit together

Ans: There 6 electronic Engineers that can be arranged in a round table in $(6-1)!$ ie, $5!$ ways.

There are 5 software engineers and they are not to sit together so we have 6 places for Software Engineers and can be placed in $6!$ ways. Therefore,



Total no. of ways to arrange the engineers
on a round table = $6! \times 5!$
 $= 120 \times 720$
 $= \underline{\underline{86400}}$

$\therefore$ Total no. of ways in which 12 people
can sit on a round table is
 $= \underline{\underline{8! \times 7!}}$

A. Determine the no. of ways in which 2 software
engineers and 6 electronic engineers
be seated at a round table so that
no two software engineers can sit

Ans: There 6 electronic engineers that can
be arranged in a round table in $(6-1)!$

Combination

A combination is a selection of some or all, objects from a set of given objects where order of the objects does not matter. The no. of combinations of n objects taken ' r ' at a time is represented by nC_r

$${}^nC_r = \frac{n!}{r!(n-r)!} \quad n \geq r \geq 1$$

No. of combinations of ' n ' things taken all at a time is ${}^nC_n = \frac{n!}{0!n!} = 1$

$${}^nC_n = 1$$

$${}^nC_1 = n$$

Examples

- (Q) How many ways can we select a software development group of 1 project leader, 5 programmers and 6 data entry operators from a group of 5 project leaders, 20 programmers and 25 data entry operators.

- Ans:
- There are 5 project leaders out of which one can be selected in 5C_1 ways
 - There are 20 programmers out of which 5 programmers can be selected in ${}^{20}C_5$ ways.
 - 25 data entry operators are there from this 6 can be selected in ${}^{25}C_6$ ways.
 - ∴ Total no. of ways to select the software development group is $= {}^5C_1 \cdot {}^{20}C_5 \cdot {}^{25}C_6$
 $= \underline{\underline{1372879200}}$

- Q. In a shipment there are 40 floppy disks of which 5 are defective. Determine
- (i) In how many ways we can select 5 floppy disks.
 - (2) In how many ways we can select 5 non defective floppy disks
 - (3) In how many ways we can select 5 floppy disks containing exactly 3 defective floppy disks?
 - (4) In how many ways we can select 5 floppy disks containing atleast 1 defective floppy disks?

(1) There are 40 floppy disks out of which³⁴ we have to select 5 floppy disks. This can be done in $40C_5$ ways.

$$\text{i.e., } 40C_5 = \frac{40!}{5! 35!} = 658008$$

(2) There are $40 - 5 = 35$ non defective floppy disks out of which we have to select 5. $\therefore$ This can be done in $35C_5$ ways.

$$\text{i.e., } 35C_5 = \frac{35!}{5! 30!} = 324632$$

(3) Total 40 floppy disks. Among this 5 are defective. We have to select exactly 3 defective from 5 in $5C_3$ ways. Remaining 2 non defectives can be selected in $35C_2$ ways.

$\therefore$ Total no. of ways to select 5 floppy disks out of which exactly 3 are defective $= 5C_3 \times 35C_2$
 $= 10 \times 595 = 5950$

(4) we have to select 5 floppy disks containing atleast one defective.

i.e., it may be 1 defective OR 2 defective. ---

OR all are defective

chances of getting one defective = $5C_1 \cdot 35C_4$

chances of getting 2 defective = $5C_2 \cdot 35C_3$

etc.

chance of getting all 5 defective = $5C_5$

$\therefore$ Total no. of chances getting atleast one defective in 5 disks

$$= 5C_1 \cdot 35C_4 + 5C_2 \cdot 35C_3 + 5C_3 \cdot 35C_2 + 5C_4 \cdot 35C_1 + 5C_5$$

$$= \underline{\underline{333376}}$$

OR

Total no. of ways to select 5 floppy disks out of 40 is $40C_5$ and

total no. of ways to select 5 floppy disks with no one defective = $35C_5$

$\therefore$ Total no. of ways to select 5 disks out of which atleast one of them is defective = $40C_5 - 35C_5$

$$= \underline{\underline{333376}}$$

Q In the DCS paper there are 8 questions
In how many ways can an examiner
select 5 questions in all if 1st question
is compulsory.

Ans Since first question is compulsory, the
examiner has to select 4 questions from
the remaining 7 questions. That can be
done in 7C_4 ways.

$$\therefore \text{No. of ways to select 5 questions} = {}^7C_4 \\ = \underline{\underline{35}}$$

Combination Problems

1. Among a set of 5 black balls and 3 red balls how many selections of 5 balls can be made such that atleast 3 of them are black balls?

Ans. Selecting atleast 3 black balls from a set of 5 black balls in a total selection of 5 balls can be

3B, 2R + 4B, 1R + 5B, 0R

total
5 Black, 3 Red

atleast 3 means
 ≥ 3

$\therefore$ Atleast 3 black balls can be selected in

$${}^5C_3 \cdot {}^3C_2 + {}^5C_4 \cdot {}^3C_1 + {}^5C_5 \cdot {}^3C_0 = \underline{\underline{46 \text{ ways}}}$$

2. How many 4 digit numbers that are divisible by 10 can be formed from the numbers 3, 5, 7, 8, 9, 0 such that no number repeats?

If a number is divisible by 10, then its unit place should contain a 0.

— — — 0

After 0 is placed in the unit place, 10's place can be filled with any of the other 5 digits.

Selecting one digit out of 5 can be done in ${}^5C_1 = 5$ ways.

After filling 10's place we have left with 4 digits. So 100's place can be

filled with ${}^4C_1 = 4$ ways.

Similarly 1000's place can be filled with remaining 3 digits.

$\therefore$ Total no. of 4 digit numbers

$$= 5 \times 4 \times 3$$

$$= \underline{\underline{60}}$$

6. In a birthday party, every person shake hand with every other person. If there was a total of 28 handshakes in the party how many persons were present in the party?

Ans: Suppose that there are n persons present in a party and every person shakes hand with every other person. Then total no. of hand shakes = $nC_2 = \frac{n(n-1)}{2}$

$$\text{Here given, } \frac{n(n-1)}{2} = 28$$

$$n(n-1) = 56$$

$$n^2 - n - 56 = 0$$

$$(n-8)(n+7) = 0$$

$$\Rightarrow n = 8, n = -7$$

$$\therefore \underline{\underline{n = 8}} \quad \underline{\underline{\text{No. of persons} = 8}}$$

Circular permutation

1. Find the number of ways in which 5 people A, B, C, D, E can be seated at a round table such that
- (i) A and B must always sit together
 - (ii) C and D must not sit together

Ans: (i) A & B must sit together, so consider this as a single unit, remaining 3 letters and this together we have 4 places. So the no. of ways arranging 4 people in a circle is, $3! = 6$ ways.

and A & B can be arranged in 2 ways

$$\therefore \text{total no. of ways} = 6 \times 2 = \underline{\underline{12}}$$

- (ii) C and D must not sit together, this can be obtained by removing all those cases (from the total possible) in which C & D are together.

The total no. of ways in which ^{people} 5 ~~people~~ can sit on a round table is $4!$

$$= 1 \cdot 2 \cdot 3 \cdot 4 = \underline{\underline{24}}$$

no. of ways in which 5 people can be arranged at a round table such that C and D together is $= 12$ ($6 \times 2 = 12$)

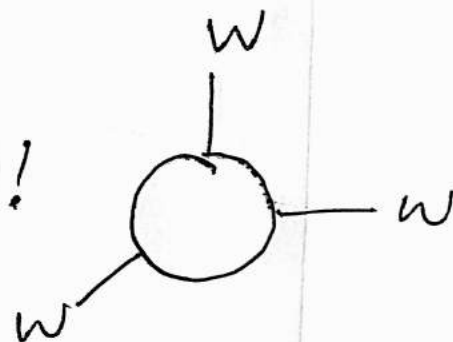
$\therefore$ required no. of ways will be

$$24 - 12 = \underline{\underline{12}}$$

⑤ In how many ways can 3 men and 3 ladies be seated at round table such that no two men are seated together?

Ans: Since we don't want the men to be seated together, the only way to do this is to make the men & women sit alternately. We will first seat the 3 women, on alternate seats.

This can be done in $(3-1)!$ i.e., $2!$ ways.



Now, the 3 men can be seated in the remaining seats in $3! = 6$ ways.

[Note that, we haven't used the formula for circular arrangements now, because after fixing the position of women, the arrangements on the remaining seats is equivalent to a linear arrangement]

$\therefore$ Total no. of ways in this case

$$= 3! \times 2! = \underline{\underline{12}}$$

Binomial theorem

The Binomial theorem is a result of expanding powers of binomials or sum of 2 terms. The coefficient of the terms in the expansion are the binomial coefficients $\binom{n}{k}$ or $n C_k$.

The theorem and its generalizations can be used to prove results and solve problems in Combinatorics, algebra, calculus and many other areas of Mathematics.

Binomial Theorem
Statement:-

Let n be a positive integer, and x and y real numbers (or complex nos or polynomials) the coefficient of $x^k y^{n-k}$ in the k^{th} term in the expansion of $(x+y)^n$ is equal to $n C_k$ where $n C_k = \frac{n!}{(n-k)! k!}$

$$\therefore (x+y)^n = \sum_{r=0}^n n C_r x^{n-r} y^r \quad \text{or} \quad \sum_{r=0}^n n C_{n-r} x^{n-r} y^r$$

$$\text{i.e., } (x+y)^n = nC_0 x^n y^0 + nC_1 x^{n-1} y^1 + nC_2 x^{n-2} y^2 + \dots + nC_k x^{n-k} y^k + \dots + nC_n x^0 y^n$$

The above expansion is known as binomial expansion.

Exa:

- Find the coefficient of x^4 in the expansion of ~~the~~ $(x+1)^9$

The coefficient of 4th term is equal to 9C_4

$$\text{i.e., } {}^9C_4 = \frac{9!}{(9-4)! 4!} = \underline{\underline{126}}$$

$\therefore$ 4th term of the expansion is, ${}^9C_4 x^4$
 i.e., $\underline{\underline{126x^4}}$

- Find the coefficient of $x^5 y^2$ in the expansion of $(x+y)^7$

Using Binomial theorem, we know

the coefficient of $x^k y^{n-k}$ is equal to nC_k

OR nC_{n-k}

here $n=7$ $k=5$

Since

$\therefore (x+y)^7$ we have to find the coefficient of $x^5 y^2$

$\therefore$ coefficient is, 7C_5 or 7C_2

$${}^7C_5 = {}^7C_2 = \underline{\underline{21}}$$

3. Find the coefficient of $a^5 b^2$ in the expansion of $(2a-3b)^7$

We know, the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$ is nC_k .

Here put $x=2a$ $y=-3b$

$\therefore$ we have to find the coefficient of $a^5 b^2$ which is, equal to,

$$(x+y)^n \Rightarrow (2a-3b)^7 \quad \begin{matrix} n=7 \\ k=5 \end{matrix}$$

$$\therefore {}^7C_5 (2a)^5 (-3b)^2$$

$$= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2^5 \times -3 \times -3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

$$= \underline{\underline{6048}} a^5 b^2$$

For positive integers n, t the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + \dots + x_t)^n$ is $\frac{n!}{n_1! n_2! n_3! \dots n_t!}$

Where each n_i is an integer with $0 \leq n_i \leq n$ for all $1 \leq i \leq t$ and $n_1 + n_2 + \dots + n_t = n$.

Exer:

① Find the coefficient of $x^2 y^2 z^3$ in the expansion of $(x+y+z)^7$

Coe. of $x^2 y^2 z^3$ in the expansion of $(x+y+z)^7$

$$\text{is} = \frac{7!}{2! 2! 3!} = \underline{\underline{210}}$$

② find the coefficient of xyz^5 in the expansion of $(x+y+z)^{10}$.

$$\text{Co. of } xyz^5 = \frac{10!}{1! 1! 5!} = \underline{\underline{30240}}$$

Combination With Repetition

Assume that we have a set of n elements. Any selection of r objects from the set where each object can be selected more than once is called a combination of n objects taken r at a time with repetition.

The formula for finding a r -combination with repetitions from ' n ' elements is

$${}^{n+r-1}C_r \quad \text{or} \quad \binom{n+r-1}{r}$$

Note: nC_k can also be represented as $\binom{n}{k}$

Note:-

The following statements are equivalent that means they have a common solution in Combinatorics.

(1.) The no. of combinations of n objects

taken 'r' at a time with repetition ⁴⁷

- (2) The no. of ways of 'r' identical objects can be distributed among 'n' distinct containers.
- (3) The no. of non negative integer solutions of the eqⁿ, $x_1 + x_2 + \dots + x_n = r$.

$$x_1 \geq 0, x_2 \geq 0, \dots, 1 \leq i \leq n.$$

In all the above 3 cases, the solution

is, $\boxed{{}^{n+r-1}C_r}$

Problems:-

1. In how many ways can one distribute 10 white marbles among 6 distinct containers?

Solving this problem is equivalent to finding the no. of non negative integer solution to the equation

$$x_1 + x_2 + \dots + x_6 = 10$$

That number is the no. of selections of size 10 with repetition, from a collection of size 6.

$$\text{ie, } n=6 \quad r=10.$$

$$\therefore \text{ Answer is, } n+r-1 C_r = {}^{15}C_{10} \\ = \underline{\underline{3003}}$$

2. Determine all integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 7$ where $x_i \geq 0$ for all $1 \leq i \leq 4$.

We have,

the no. of integer solutions of the equation

$$x_1 + x_2 + \dots + x_n = r, \quad x_i \geq 0 \quad 1 \leq i \leq n$$

is equal to $\boxed{n+r-1 C_r}$

$$\text{here } n=4, \quad r=7$$

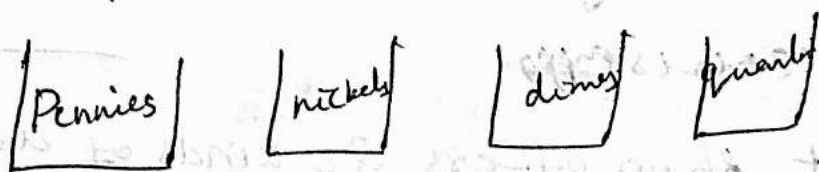
$$\text{no. of solutions} = {}^{4+7-1}C_7 = {}^{10}C_7$$

$$= \underline{\underline{120}}$$

One can find out the solns of this eqⁿ randomly.

4. Determine how many ways 20 coins can be selected from 4 large containers filled with pennies, nickels, dimes and quarters. (Each container is filled with only one type of coin)

We have, 4 containers.



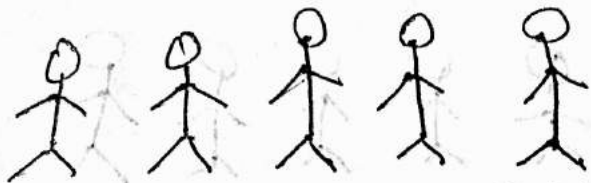
$n = 4$ We have to select 20 coins from these boxes. $r = 20$.

$\therefore$ no. of ways of selecting 20 objects repeatedly from 4 containers

$$= \underline{\underline{23C_{20}}} = \underline{\underline{1771}} \quad n+r-1C_r$$

- 5 In how many ways can 10 chocolates be distributed among 5 children if
- there are no restrictions
 - each child gets atleast one chocolate
 - the oldest child gets atleast 2 chocolates.

(a) There are 5 children.



$$n = 5$$

$$r = 10$$

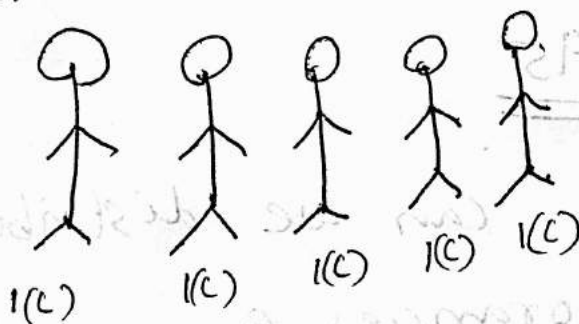
$\therefore$ no. of ways of distributing 10 chocolates among 5 children =

$${}^{10+5-1}C_{10}$$

$$= {}^{14}C_{10} = 1001$$

(b) each child ^{should} gets at least one chocolate

means we are ~~giving~~ ^{giving} one chocolate compulsory to each child. So there remains 5 chocolates.



remaining

$$n = 5$$

$$r = 5$$

So no. of ways of distributing 5 chocolates

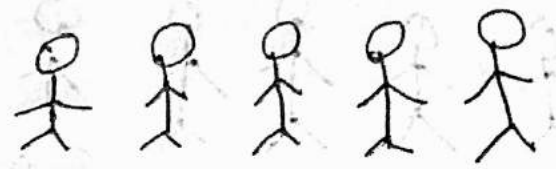
among 5 children =

$${}^{5+5-1}C_5$$

$$= {}^9C_5 = 126$$

(c) the oldest child gets at least 2 chocolates.

Let A be the oldest-child among 5 children.



A(2).

the oldest child A should get at least 2 chocolates. so let us fix that.
(A may get more than two).

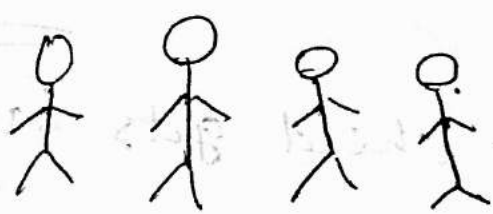
Now remaining 8 chocolates are there and we have to distribute these among 5 children, that can be

done in ${}^{5+8-1}C_8$ $n=5$
 $r=8$
 $= 12 C_8$ ways.
 $= \underline{\underline{495}}$

6. In how many ways can we distribute 7 apples and 6 oranges among 4 children so that each child receives at least one apple.

We have 7 apples & 6 oranges.

$n=4$.



$(1A) \quad (1A) \quad (1A) \quad (1A)$

each child should receive at least one apple. So let us fix that remaining 3 apples can be distributed among 4 children in $\binom{4+3-1}{3} = {}^6C_3 = 20$ ways.

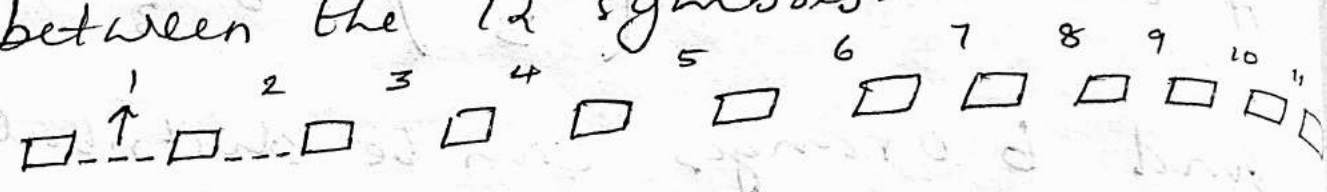
And 6 oranges can be distributed among 4 children in $\binom{6+4-1}{6} = {}^9C_6 = 84$ ways.

So by the rule of product there are

$20 \times 84 = 1680$ ways to distribute the fruits under the given condition

7. A message is made up of 12 different symbols and is to be transmitted through a communication channel. In addition to the 12 symbols, the transmitter will also send a total of 45 (blank) spaces between the symbols, with at least 3 spaces between each pair of consecutive symbols. In how many ways can the transmitter send such a message?

There are $12!$ ways to arrange the 12 different symbols and for each of these arrangements there are 11 positions between the 12 symbols.



Because there must be atleast 3 spaces between successive symbols we use $11 \times 3 = 33$ of the 45 spaces.

Now we have $45 - 33 = 12$ remaining spaces. This is now a selection, with repetition of size 12 from a collection of 11 positions.

ie, we have to distribute 12 spaces in to 11 positions. and this can be done in $C(11+12-1, 12) = {}^{23}C_{12}$ ways.

$\therefore$ By the rule of product the transmitter can send such messages in $12! \times {}^{23}C_{12} = 3.097 \times 10^{14}$ ways

8. Determine the no. of integer solution 54
for $x_1 + x_2 + x_3 + x_4 + x_5 < 40$ where

(i) $x_i \geq 0 \quad 1 \leq i \leq 5$

(ii) $x_i \geq -3 \quad 1 \leq i \leq 5$

Ans. We have to find the integer solution
for the inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 < 40 \quad \text{--- (1)}$$

We can rewrite this inequality to
equality by adding one more variable
into it. i.e.,

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 40 \quad \text{--- (2)}$$

where $x_6 \geq 0$

i.e., $x_6 \geq 1$ (x_6 is an integer)

$$\therefore x_6 - 1 \geq 0$$

$$\text{(2)} \Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 40$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 - 1 = 40 - 1$$

(Subtracting 1 from both sides) (3)

Now let $y_i = x_i$ for $1 \leq i \leq 5$ & $y_6 = x_6 + 1$

$$\therefore \textcircled{3} \Rightarrow y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 39 \quad \textcircled{4}$$

Where $y_i \geq 0$.

Now the no. of integer soln of $\textcircled{1}$ equivalent to the no. of integer soln of $\textcircled{4}$.

$$\begin{aligned} \text{no. of solution of } \textcircled{4} &= (6 + 39 - 1) C_{39} \\ &= 44 C_{39} \\ &= \underline{\underline{1,086,008}} \end{aligned} \quad \left| \begin{array}{l} n=6 \\ r=39 \\ n+r-1 \end{array} \right.$$

(b) $x_i \geq -3 \quad 1 \leq i \leq 5$

$$\textcircled{1} \Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 < 40$$

$$\Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 40 \quad \textcircled{2}$$

Where we have 2 conditions.

$$\left. \begin{array}{l} 1) \quad x_6 \geq 0 \\ \text{or } x_6 \geq 1 \\ x_6 - 1 \geq 0 \end{array} \right\} \begin{array}{l} 2) \quad x_i \geq -3 \\ x_i + 3 \geq 0 \end{array}$$

Now eqⁿ $\textcircled{2}$ becomes,

$$\begin{aligned} (x_1 + 3) + (x_2 + 3) + (x_3 + 3) + (x_4 + 3) + (x_5 + 3) + (x_6 - 1) \\ = 40 + 15 - 1 \end{aligned}$$

Let $y_i = x_i + 3$, $1 \leq i \leq 5$ and $y_6 = x_6 - 1$
 then we get,

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 54 \rightarrow \textcircled{5}$$

each $y_i \geq 0$.

Soln of $\textcircled{1} = \text{soln of } \textcircled{5}$

$$\therefore \text{soln of } \textcircled{5} = (6 + 54 - 1) C_{54}$$

$$= 59 C_{54}$$

$$\left| \begin{array}{l} n=6 \\ r=54. \\ n+r-1 C_r \end{array} \right.$$

The Pigeon hole Principle.

Statement:-

If m pigeons occupy n pigeon holes and $m > n$ then there is at least one pigeonhole that contains two or more pigeons.

Generalisation of the above principle:-

It states that if m pigeons are assigned to n pigeon holes and the no. of pigeons is very large than the no. of holes then one of the pigeon must contain at least $\left\lfloor \frac{m-1}{n} \right\rfloor + 1$ pigeons.

Problems: -

1. S.T in a group of 8 people atleast 2 have birthdays which fall on the same day of the week in a given year.

Let 8 people as the 8 pigeons and 7 days of a week as the 7 pigeonholes.

Then by pigeon hole principle,

$$\text{atleast } \left[\frac{m-1}{n} + 1 \right] = \left[\frac{8-1}{7} + 1 \right] \quad \begin{array}{l} m=8 \\ n=7 \\ m>n \end{array}$$

= 2 will have birthday on the same day.

- (2) S.T atleast 2 people must have their birthday on the same month if 13 people are assembled in a room.

Consider 13 people as 13 pigeons and 12 months in a year as 12 pigeonholes. Then by pigeon hole principle

$$\text{atleast } \left[\frac{13-1}{12} + 1 \right] = 2 \text{ will have birthday on same month. } \begin{array}{l} m=13 \\ n=12 \end{array} \left[\frac{m-1}{n} + 1 \right]$$

show that
(3) If 9 colours are used to paint 100⁵⁹ houses, then atleast 12 houses will be of the same colour.

Let us assume, the colours as the pigeon holes and the houses as pigeons

so using pigeon hole

$$m = 100$$

$$n = 9 \therefore m > n$$

principle, there is

$$\text{atleast } \left[\frac{m-1}{n} + 1 \right] = \left[\frac{100-1}{9} + 1 \right] = 12 \text{ houses}$$

will be of the same colour.

4. 7 members of family have total Rs. 2886 in their pockets. Show that atleast one of them must have Rs. 416 in his pocket.

Assume that 7 members as 7 pigeon holes and the rupees as pigeons.

$$m = 2886$$

so by P. hole principle,

$$n = 7$$

$$\left[\frac{2886-1}{7} + 1 \right] = 416 \text{ rupees will be in}$$

one member's pocket.

Questions:-

1. What is the minimum no. of students required in an English class to be sure that atleast 6 will receive the same grade, if there are 5 possible grades.

Consider no. of students as 'pigeons' and grades as pigeon holes $n = 5$.

So by P-hole property there are, atleast

$$\left[\frac{m-1}{n} + 1 \right] = \left[\frac{m-1}{5} + 1 \right] = 6 \quad \left[\text{people will have same grade.} \right]$$

$$\frac{m-1}{5} = 5$$

$$m-1 = 25$$

$$m = \underline{\underline{26}}$$

So the min no. of students = 26

Principle of Inclusion & Exclusion.

Theorem:-

Consider a set S , with $|S| = N$ and conditions C_i , $1 \leq i \leq t$ satisfied by some of elements of S . The no. of elements of S that satisfy ^{none of} the conditions C_i is denoted by $\bar{N} = N(C_1 \bar{C}_2 \bar{C}_3 \dots \bar{C}_t)$

Where

$$\begin{aligned} \bar{N} = & N - [N(C_1) + N(C_2) + \dots + N(C_t)] + \\ & [N(C_1 C_2) + N(C_1 C_3) + \dots + N(C_1 C_t) + N(C_2 C_3) + \dots \\ & \quad N(C_{t-1} C_t)] \\ & - [N(C_1 C_2 C_3) + N(C_1 C_2 C_4) + \dots + N(C_1 C_2 C_t) + N(C_1 C_3 C_4) + \dots \\ & \quad + N(C_1 C_3 C_t) + \dots + N(C_{t-2} C_{t-1} C_t)] + \dots + (-1)^t N(C_1 C_2 \dots C_t) \end{aligned}$$

Note:- ⁽¹⁾

Under the hypotheses of the above theorem the no. of elements in S that satisfy at least one of the conditions C_i where $1 \leq i \leq t$ is $N(C_1 \text{ or } C_2 \text{ or } \dots \text{ or } C_t) = N - \bar{N}$

(2) We are using some notations for simplifying the statement of the above theorem. We write,

$$S_0 = N$$

$$S_1 = [N(C_1) + N(C_2) + \dots + N(C_t)]$$

$$S_2 = [N(C_1 C_2) + N(C_1 C_3) + \dots + N(C_1 C_t) + N(C_2 C_3) + \dots + N(C_{t-1} C_t)]$$

In general

$$S_k = \sum N(C_1 C_2 \dots C_k), \quad 1 \leq k \leq t$$

Problems:-

1. Determine the no. of positive integers n where $1 \leq n \leq 100$ and n is not divisible by 2, 3 or 5.

Here $S = \{1, 2, 3, \dots, 100\}$ $N = 100$

For $n \in S$, n satisfies

- (a) Condition C_1 , if n is divisible by 2
- (b) Condition C_2 , if n is divisible by 3
- (c) Condition C_3 , if n is divisible by 5

Then the answer to this problem is,

$$N(\bar{C}_1 \bar{C}_2 \bar{C}_3)$$

Let $[x]$ denote the greatest integer less than or equal to x for any real x .

$$\therefore N(C_1) = \left[\frac{100}{2} \right] = 50 \quad \left[\text{there are } 2, 4, 6, \dots, 98, 100 \text{ i.e., } 50 \text{ integers which are divisible by } 2. \right]$$

$$N(C_2) = \left[\frac{100}{3} \right] = \left[33\frac{1}{3} \right] = 33$$

there exist 33 integers which are divisible by 3.

$$N(C_3) = \left[\frac{100}{5} \right] = 20 \quad \text{integers which are divisible by 5}$$

$$N(C_1 C_2) = \left[\frac{100}{2 \times 3} \right] = 16 \quad \left[\text{Take the L.C.M of } 2 \text{ \& } 3. \right]$$

$$N(C_2 C_3) = \left[\frac{100}{3 \times 5} \right] = 6.$$

$$N(C_1 C_3) = \left[\frac{100}{2 \times 5} \right] = 10.$$

$$N(C_1 C_2 C_3) = \left[\frac{100}{2 \times 3 \times 5} \right] = 3$$

Applying the inclusion & exclusion principle

$$N(\overline{C_1} \overline{C_2} \overline{C_3}) = S - S_1 + S_2 - S_3$$

$$= N - [N(C_1) + N(C_2) + N(C_3)] + [N(C_1 C_2) + N(C_2 C_3) + N(C_1 C_3)]$$

$$= N(C_1 C_2 C_3)$$

$$= 100 - [50 + 33 + 20] + [16 + 10 + 6] - 3$$

$$= \underline{\underline{26}}$$

There are 26 integers in S which are not divisible by 2, 3 or 5.

(2) How many solutions does $x_1 + x_2 + x_3 = 11$ have ^{where} x_1, x_2 and x_3 are non negative integers with $x_1 \leq 3, x_2 \leq 4$ and $x_3 \leq 6$?

★ ★

Previous Result:-

No. of integer soln of $x_1 + x_2 + \dots + x_n = r,$

$x_i \geq 0, 1 \leq i \leq n$ is equal to $\boxed{n+r-1 \choose r}$

Here we have to find the solns of

$x_1 + x_2 + x_3 = 11$ with the conditions

$x_1 \leq 3, x_2 \leq 4, x_3 \leq 6.$

To apply the principle of inclusion-exclusion, let a solutions have property C_1 if $x_1 > 3$.

solution have property C_2 if $x_2 > 4$

solutions have property C_3 if $x_3 > 6$.

Then the no-of solutions using the inequalities $x_1 \leq 3, x_2 \leq 4, x_3 \leq 6$ is $N(\bar{C}_1 \bar{C}_2 \bar{C}_3)$

$$\therefore N(\bar{C}_1 \bar{C}_2 \bar{C}_3) = S_0 - S_1 + S_2 - S_3 \quad \text{--- (1)}$$

$$\Rightarrow N = [N(C_1) + N(C_2) + N(C_3)] + [N(C_1 C_2) + N(C_1 C_3) + N(C_2 C_3)] - N(C_1 C_2 C_3)$$

• N = the total no. of solutions with $x_1, x_2, x_3 \in \mathbb{N}$

$$= \binom{11+3-1}{11} = {}^{13}C_{11} = 78$$

• $N(C_1)$ = no. of solutions with $x_1 > 3$

so $(x_1 \geq 4) \Rightarrow \binom{7+3-1}{7} = \binom{9}{7} = 36$

• $N(C_2) = \text{no. of solutions with } x_2 \geq 4$

$$\text{So } (x_2 \geq 5) = \binom{6+3-1}{6} = 8 C_6 = \underline{\underline{28}}$$

• $N(C_3) = \text{no. of solutions with } x_3 \geq 6$

$$\text{So } (x_3 \geq 7) = \binom{4+3-1}{4} = 6 C_4 = \underline{\underline{15}}$$

• $N(C_1 C_2) = \text{no. of solns with } x_1 \geq 4 \text{ and}$

$$x_2 \geq 5 \Rightarrow \binom{2+3-1}{2} = 4 C_2 = \underline{\underline{6}}$$

• $N(C_1 C_3) = \text{no. of solns with } x_1 \geq 4 \text{ } x_3 \geq 7$

$$= \binom{0+3-1}{0} = 2 C_0 = \underline{\underline{1}}$$

• $N(C_2 C_3) = \text{no. of solns with } x_2 \geq 5 \text{ and } x_3 \geq 7$

$$= \underline{\underline{0}}$$

• $N(C_1 C_2 C_3) = \text{no. of solns with } x_1 \geq 4,$

$$x_2 \geq 5 \text{ and } x_3 \geq 7 = \underline{\underline{0}}$$

eqn (1) becomes.

$$\therefore N(\bar{C}_1 \bar{C}_2 \bar{C}_3) = 78 - 36 - 28 - 15 + 6 + 1 + 0 - 0$$

$$= \underline{\underline{6}}$$

Derangements

We may use the principle of inclusion exclusion to count the permutation of n objects that leave no objects in their original position.

Exa: A new employee checks the hats of n people at a restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the probability that no one receives the correct hat? (The Hat-check problem)

The ans is,

the no. of ways the hats can be arranged so that there is no hat in its original position, divided by $n!$ (the no. of permutations of n hats)

Defn:- A derangement is a permutation of objects that leaves no object in their original positions.

Let D_n denote the no. of derangements of n objects. Then,
the no. of derangements of a set with ' n ' objects is,
$$D_n = n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!} \right]$$

Exa:-

- 1- The number of derangements of 1, 2, 3, 4 is,

$$D_4 = 4! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \right]$$

$$= 4! \left[\frac{1}{2} - \frac{1}{6} + \frac{1}{24} \right]$$

$$= 4! \times \frac{9}{24} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \times 9}{24} = \underline{\underline{9}}$$

Principle of Exclusion & Inclusion

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1. Among 18 students in a room 7 study mathematics 10 study science and 10 study Computer programming. Also 3 study M & S, 5 study S & C, 4 study M & C, 1 student studies all 3 subjects. How many of these students study none of the 3 subjects?

Here $N = 18$

Let C_1 : Students studying Maths

C_2 : Students studying Science

C_3 : Computer Science

$$\text{then } N(C_1) = 7$$

$$N(C_2) = 10$$

$$N(C_3) = 10$$

$$N(C_1 \cap C_2) = 3$$

$$N(C_2 \cap C_3) = 5$$

$$N(C_1 \cap C_3) = 4$$

$$N(C_1 \cap C_2 \cap C_3) = 1$$

$$\therefore N(\bar{C}_1 \bar{C}_2 \bar{C}_3) = \text{no. of students study none of the subject}$$

$$\Rightarrow S_8 - S_1 + S_2 - S_3$$

$$\Rightarrow 18 - [7 + 10 + 10] + [5 + 4 + 3] - 1$$

$$\Rightarrow 18 - 27 + 12 - 1$$

$$\Rightarrow \underline{\underline{2}}$$

2. A survey of 550 television watchers produced the following information: 285 watch football games, 195 watch hockey games, 115 watch base balls, 45 watch football and base ball games, 70 watch football & hockey games, 50 watch hockey and base ball games, 100 do not watch any of 3 games.

(a) How many people in the survey watch all 3 games?

(b) How many people watch exactly one of the games.

Here $N = 550$

Let C_1 : no. of persons watching football
i.e., $N(C_1) = 285$

C_2 : no. of persons watching hockey = 195
 $N(C_2) = 195$

S_3 : ^{no. of persons} watching base balls $N(S_3) = 115$ 71

$$N(C_1, C_3) = 45 \quad N(C_1, C_2) = 70 \quad N(C_2, C_3) = 50$$

$$N(\bar{C}_1 \bar{C}_2 \bar{C}_3) = 100 \quad (\text{no. of people not watching any of 3 games})$$

$\therefore$ N. of people who watch all 3 games
 $= N(C_1 C_2 C_3)$.

We have,

$$\begin{aligned} N(\bar{C}_1 \bar{C}_2 \bar{C}_3) &= S_0 - S_1 + S_2 - S_3 \\ &= N - [N(C_1) + N(C_2) + N(C_3)] + \\ &\quad [N(C_1 C_2) + N(C_2 C_3) + N(C_1 C_3)] - N(C_1 C_2 C_3) \end{aligned}$$

$$\begin{aligned} \therefore N(C_1 C_2 C_3) &= 550 - [285 + 195 + 115] + [45 + 70 + 50] - 100 \\ &= \underline{\underline{20}} \end{aligned}$$

$\therefore$ 20 people watch all the 3 games.

(b) 20 people watch all the 3 games.

$\therefore$ people who watch football only

Defn:- A derangement is a permutation of objects that leaves no object in their original positions.

Let D_n denote the no. of derangements of n objects. Then,

the no. of derangements of a set with ' n ' objects is,

$$D_n = n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!} \right]$$

Exa:-

- 1- The number of derangements of 1, 2, 3, 4 is,

$$D_4 = 4! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \right]$$

$$= 4! \left[\frac{1}{2} - \frac{1}{6} + \frac{1}{24} \right]$$

$$= 4! \times \frac{9}{24} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \times 9}{24} = \underline{\underline{9}}$$

Note:- We denote Derangement of n element as D_N

D_N = Total permutations without any restriction

no. of permutation in which atleast one element is in its original position.

$$\text{ie, } D_N = n! - n(A_1 \cup A_2 \cup \dots \cup A_n)$$

Note:- 2 From principle of inclusion & exclusion
we have Let $A_i =$ Set of permutations in which
i-th element is in its original position

$$n(A_1 \cup A_2 \cup \dots \cup A_n) = \sum_{(nC_1 \text{ terms})} n(A_i) - \sum_{(nC_2 \text{ terms})} n(A_i \cap A_j)$$

$$+ \sum_{nC_3} n(A_i \cap A_j \cap A_k) - \dots - (-1)^{n+1} n(A_1 \cap A_2 \cap \dots \cap A_n)$$

Using these result, we get

$$D_N = n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^{n+1} \frac{1}{n!} \right]$$

Problems

74

- Find
1. The no. of ways in which all the letters of the word SWORD can be arranged such that no letter is in its original position.
 2. There are 5 balls of different colours and 5 boxes of same colours as those of balls. Find the no. of ways in which the balls on in each box could be placed such that no ball goes to box of its own colour.
 3. Let $A = \{x_1, x_2, x_3, x_4, x_5\}$; $B = \{y_1, y_2, y_3, y_4, y_5\}$
function is defined from set A to B.
Find the no. of 1-1 function such that
 $f(x_i) \neq y_i$ for $i = 1, 2, 3, 4, 5$

Ans:-

All the 3 questions are examples of a derangement problem.

In the 1st, can we have to arrange the letters of the word "SWORD" in such a manner that no letter is in

it's original position.

That means, S will not be in 1st position,
W will not be in 2nd position like that.

So this is a problem of derangement
of 5 distinct letters.

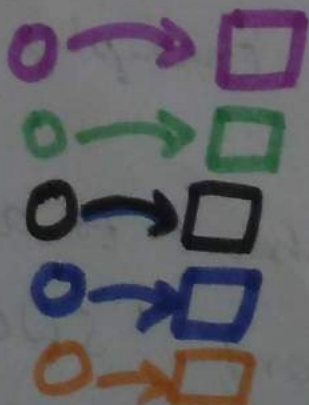
i.e., we have to find the no. of permutations
of 5 letters that leave no letter in
their original position.

$$\therefore D_5 = 5! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \right]$$

$$= 5! \left[\frac{60 - 20 + 5 - 1}{5!} \right]$$

$$= \underline{\underline{44}}$$

(2) In the 2nd case, there are 5 balls
of different colours & 5 boxes of same
colours.



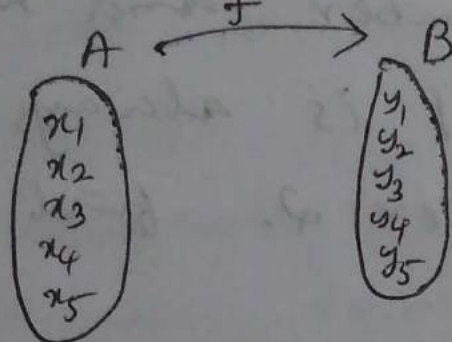
Here we have to consider the case

on which, each ball should not go to the box of same colour. 76

so clearly it is the case of Derangement of 5 objects.

$$\therefore D_5 = \underline{\underline{44}}$$

(3) In the 3rd case, we have 2 sets.



Here no. of 1-1 function is $5!$.
(distinct element of A is mapped into a distinct element of B which is unique)

We can define, $5!$ 1-1 functions from A to B .

But here we have to consider the 1-1 function in which $f(x_i) \neq y_i$

for $i = 1, 2, 3, 4, 5$

i.e., x_1 is not mapped to y_1 ,

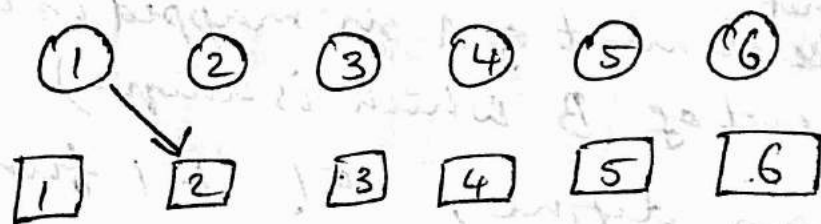
x_2 is not mapped to y_2 like that

So this is the case of derangement of 5 objects. i.e., the answer is

$$D_5 = 5! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \right] = \underline{\underline{44}}$$

Q) 6 cards and 6 envelopes are numbered 1, 2, 3, 4, 5, 6 and cards are to be placed in envelopes so that each envelope contains exactly one card and no card is placed in the envelope bearing the same number and moreover card number 1 is always placed in envelope number 2. find no. of ways.

We have 6 cards.



Card 1 should be placed in envelope 2.

This is the case of selective derangement
first we consider the derangement of 6 object i.e., D_6

$$D_6 = 6! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} \right)$$

$$D_6 = 265$$

In these derangements, we have

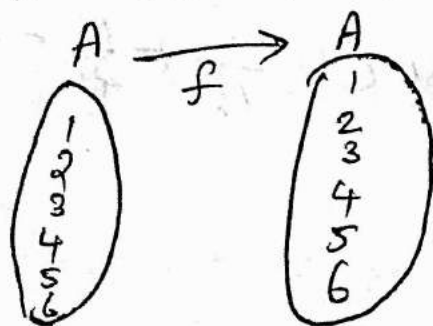
the cases in which card ① is placed in 2, card ② is placed in envelope ③, ③ is placed in envelope 4, like that it, here we have 5 different cases. But we are considering only one, case is, card 1 is placed in envelope 2.

Also in each cases, the no. of derangements are same.

$\therefore$ the no. of ways in which card 1 is placed in envelope number 2 is,

$$= \frac{2 \cdot 6 \cdot 5}{5} = \underline{\underline{6 \cdot 5 \cdot 3}}$$

⑧ Let $f: A \rightarrow A$ be an invertible function where $A = \{1, 2, 3, 4, 5, 6\}$. The no. of these functions in which atleast 3 elements have self image.



f is invertible if it is 1-1 & onto.

the no. of 1-1 functions from $A \rightarrow A$ is $6!$. We want the no. of functions in which at least 3 have self image.

exa.

$$\begin{aligned} f(1) &= 1 \\ f(2) &= 2 \\ f(3) &= 3 \end{aligned}$$

at least 3 have self image means that we may have 4 self images or may have 5 self images or may have 6 self images.

$\therefore$ no. of fns in which exactly 3 elements have self image $= {}^6C_3 \times 2$

[3 elements can be selected from 6 in 6C_3 ways. and if we have 3 elements having self image then remaining there exist a derangement of 3 elements.

$$\text{So } D_3 = 3! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \right) = \underline{\underline{2}}$$

Similarly
no. of fns in which 4 elements have
self image = ${}^6C_4 \times D_2$

$$D_2 = 2! \left(1 - \frac{1}{1!} + \frac{1}{2!}\right)$$

$$= {}^6C_4 \times 1$$

no. of fns in which 5 elements
have self image

$$= {}^6C_5 \times 0$$

no. of fns in which all elements have
self image = 6C_6

$\therefore$ no. of fns in which atleast 3 have

$$\text{self image} = {}^6C_3 \times 2 + {}^6C_4 \times 1 + {}^6C_5 \times 0 + {}^6C_6$$

$$= \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} \times 2 + \frac{6 \cdot 5}{1 \cdot 2} \times 1 + 0 + 1$$

$$= 40 + 15 + 1$$

$$= \underline{\underline{56}}$$

- ⑦ Find the no. of arrangements of all the
digits of 12345 such that atleast
3 digits will not come in it's
position.

Here the digits are 1, 2, 3, 4, 5.

3 digits will not come in its position means we have to consider the

derangement of 3 distinct integers.

Now,

at least 3 digits will not come means,

it may contain D_3 (derangement of 3 digits), ^{or} it may contain D_4 (derangement of 4 digits) or it may contain D_5 .

$\therefore$ solution is, $D_3 + D_4 + D_5$

$$D_3 = {}^5C_3 \times 2! \text{ (remaining 2 letters } D_2) \quad \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ & & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & 5 \end{array}$$

(3 digits can be selected from 5 in 5C_3 ways)

$$D_4 = {}^5C_4 \times 4! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$

$$D_5 = {}^5C_5 \times 5! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$$

$$\therefore D_3 + D_4 + D_5 = 20 + 45 + 44$$

$$= \underline{\underline{109}}$$

Module I Fundamentals of Counting Theory

Rule of Sum & Rule of Product

The Rule of Sum and the rule of product are two basic principles of counting that are used to build up the theory and understanding of enumerative combinatorics.

Rule of Sum :- (statement)

If there are ' m ' choices for one action and ' n ' choices for another action and the 2 actions cannot be done at the same time, then there are ' $m+n$ ' ways to choose one of these actions.

Rule of Product :- (statement)

If there are ' m ' ways of doing something and ' n ' ways of doing another thing after that, then there are ' $m \times n$ ' ways to perform both of these actions.

Exa:-

- ① During a local campaign 8 Republican and 5 democratic candidates are nominated for

President of the school board.

(a) If the President is to be one of these candidates, how many possibilities are there for the eventual winner?

(b) How many possibilities exist for a pair of candidates (one from each party) to oppose each other for the eventual election?

Ans:

(a) There are 8 Republican and 5 democratic candidates. Hence the no. of possibilities for the eventual winner is,

$$5 + 8 = 13$$

by the rule of Sum, we can select one from the given candidate by 13 ways.

(b) We have to select a pair of candidate one from each party.

$\therefore$ That we can select by 5×8 ways.

[One person from 8 Republican & one person in 8 ways] from democratic in 5 ways

$\therefore$ As a pair we can select it by $5 \times 8 = 40$ ways. [Rule of product]

(2) Buick automobiles come in 4 models, 12 colours, 3 engine sizes and 2 transmission types

(a) How many distinct Buicks can be manufactured

(b) If one of the available colours is blue, how many different ^{Blue} Buicks can be manufactured

Ans. (a) Buick comes in 4 models, 12 colours, 3 engine sizes and 2 transmission type
 $\therefore$ no. of distinct Buicks can be manufactured

$$\text{in } 4 \times 12 \times 3 \times 2 = 288 \text{ ways.}$$

(b) If one of the available colour is blue, then we can select it by,

$$4 \times 1 \times 3 \times 2 = 24$$

$\therefore$ no. of different blue

Buicks can be manufactured

$$\text{is, } \underline{\underline{= 24}}$$

{ we count only blue colour vehicles.
 $\therefore$ only 1 colour i.e., blue) will be taken

Fundamentals of LogicSection 2.1Basic Connectives and Truth tables:-Proposition:-

A declarative sentence which is true or false, but not both is called a proposition or (statements.) We use the lower case letters of the alphabet such as P, q and r to represent the statements

Exa:-

P : The integer 5 is a prime number

q : Addition of 2 and 3 gives 4 as their sum

r : Today is Monday.

Here the proposition P has the truth value 'true' (ie, 1) and q has the truth value 'false' (ie 0)

The truth value of ' r ' depends upon the day in which the statement is made.

Note:- We do not regard sentences such⁸⁶ as the exclamation or the command as statements

ex: "What a beautiful evening!"

"Get up and do your exercises"

Connectives:-

1. Negation:-

If 'P' is a statement then its negation is $\sim P$ or $\neg P$ read as 'not P'

Truth table of negation

P	$\neg P$
1	0
0	1

OR

P	$\neg P$
T	F
F	T

2. Conjunction:-

Let P and Q be 2 propositions. The proposition

$P \wedge Q$ is called conjunction of P and Q read as P and Q.

The statement $P \wedge Q$ has a truth value 1 when both P and Q have the truth value 1 and is assigned to truth value 0 otherwise.

Truth table of Conjunction $P \wedge Q$

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P	Q	$P \wedge Q$
1	0	0
0	1	0
1	1	1
0	0	0

3. Disjunction

If P and Q are 2 statements, then the statement $P \vee Q$ is called a disjunction read as P OR Q .

The statement $P \vee Q$ has the truth value 0 only when both P and Q have the truth value 0, otherwise it is 1.

Truth table of disjunction $P \vee Q$

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

Conditional Statement (Implication)

If P and q are any 2 statements, then the statement " $P \rightarrow q$ " which is read as "if P then q " is called a Conditional Statement.

The statement $P \rightarrow q$ has a truth value 1 (true) in all cases except where P is 1 and q is 0.

Truth table for $P \rightarrow q$

P	q	$P \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

Biconditional Statement

If P and q are any 2 statements, then the statement $P \leftrightarrow q$ which is read as " P iff q " is called biconditional Statement.

The statement $P \leftrightarrow q$ has the truth value 1 (True) when both P and q have identical truth values otherwise it is 0 (Zero).

Truth table for $P \leftrightarrow Q$

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P	Q	$P \leftrightarrow Q$
1	0	0
0	1	0
1	1	1
0	0	1

Problems:-

1- Which of the following are statements.

(a) Is 2 a positive number?

(b) $x^2 + x + 1 = 0$

(c) study logic

(d) There will be snow in January

(e) If stock prices fall then I will lose money.

Ans:-

(a) not a statement. (it is a question)

(b) It is a statement (it is a declarative sentence and have truth values depending upon the values of x .)

(c) It is a command so not a statement

(d) is a declarative sentence whose truth or falsity we don't know at this time. however we can in principle determine if it is true or false /

So it is a statement.

(e) a statement. (It has a true value either true or false)

2. Give the negation of following statements. 90

(a) $P: 2+3 > 1$

(b) $q: \text{It is cold}$

(c) $r: \text{It will rain tomorrow or it will snow tomorrow}$

(d) $s: \text{If you drive, then I will walk.}$

Ans:- (a) $\neg P: 2+3$ is not greater than 1

or, $\neg P: 2+3 \leq 1$

If P is true then $\neg P$ must be false.

(b) $\neg q: \text{It is not cold.}$

(c) $\neg r: \text{It will not rain tomorrow and it will not snow tomorrow.}$

(d) $\neg s: \text{It is not the case that if you drive, I will walk.}$

3. Converting English sentences using connectives

Examples.

1. If it rains, then I will stay home

2. He is poor but honest.

3. neither it is hot nor cold today

4. Birds fly iff sky is clear
5. It is false that he is poor but not honest
6. Either today is Sunday or Monday
7. We will leave whenever he comes
8. I will not go to class unless you come

Ans:

1. Let P : It rains, Q : I will stay at home

If P then Q

$\therefore$ Symbolic form is, $P \rightarrow Q$

2. P : He is poor, Q : he is honest
replace "but" with "and"

$\therefore P \wedge Q$

3. P : It is hot today, Q : It is cold today
"neither P nor Q " can be replaced as
Not P and not Q

$\therefore \neg P \wedge \neg Q$

4. P : Birds fly, Q : Sky is clear
Symbolic,
 $\therefore P \Leftrightarrow Q$

5. P : He is poor Q : He is honest

replace "but" with "and"

"It is false that he is poor and not honest"

$$\sim(P \wedge \sim Q)$$

6. P : Today is Sunday Q : Today is Monday

It can be re-written as "Today is Sunday or Monday."

Symbolic form is $P \vee Q$

7. We can replace "whenever" with "if"

P : He comes

Q : We will leave

$$\therefore P \rightarrow Q$$

8. P : I will go to class

Q : You come

here P unless Q

$$\text{ie, } \sim Q \rightarrow P.$$

Word	Replacement															
And	conjunction ($\wedge$)															
Or	disjunction ($\vee$)															
But	and															
Whenever	if															
When	if															
either P or Q	$P \vee Q$															
neither P nor Q	not P and not Q															
P unless Q	$\sim Q \rightarrow P$															
If P then Q	$P \rightarrow Q$ <table border="1"> <tr><td>P</td><td>Q</td><td></td></tr> <tr><td>1</td><td>1</td><td>1</td></tr> <tr><td>1</td><td>0</td><td>0</td></tr> <tr><td>0</td><td>1</td><td>1</td></tr> <tr><td>0</td><td>0</td><td>1</td></tr> </table>	P	Q		1	1	1	1	0	0	0	1	1	0	0	1
P		Q														
1		1	1													
1		0	0													
0		1	1													
0	0	1														
P is sufficient for Q																
Q is necessary for P																
P only if Q																
Q if P																

Note: While solving questions, we can replace

• $P \rightarrow Q$ with $\sim P \vee Q$

• $P \rightarrow Q$ is equivalent to $\sim Q \rightarrow \sim P$

Tautology & Contradiction

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A compound propositional statement is called Tautology if it is true for all truth values of a compound statement and it is called a contradiction if it is false for all such assignments.

(a) Construct the truth table for $P \vee \neg Q$

P	Q	$\neg Q$	$P \vee \neg Q$
1	1	0	1
1	0	1	1
0	1	0	0
0	0	1	1

(b) Construct the truth table $(P \vee Q) \vee \neg P$

P	Q	$P \vee Q$	$\neg P$	$(P \vee Q) \vee \neg P$
1	1	1	0	1
1	0	1	0	1
0	1	1	1	1
0	0	0	1	1

- Q. Construct the truth table for $(P \rightarrow Q) \vee (Q \rightarrow P)$
check whether the statement is a tautology.

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \vee (Q \rightarrow P)$
1	0	0	1	1
1	1	1	1	1
0	1	1	0	1
0	0	1	1	1

All the truth values of $(P \rightarrow Q) \vee (Q \rightarrow P)$ are true.

$\therefore$ It is a tautology.

- Q. Construct the truth table of $(P \vee Q) \wedge (Q \vee R)$

P	Q	R	$P \vee Q$	$Q \vee R$	$(P \vee Q) \wedge (Q \vee R)$
1	0	0	1	0	0
1	0	1	1	1	1
1	1	0	1	1	1
0	1	0	1	1	1
0	0	1	0	1	0
0	1	1	1	1	1
1	1	1	1	1	1
0	0	0	0	0	0

H.W.

- ① Construct the truth table for following:-

(a) $(P \vee Q) \rightarrow (P \wedge Q)$

(b) $(P \rightarrow Q) \rightarrow (Q \rightarrow P)$

(c) $(Q \rightarrow \neg P) \leftrightarrow (P \leftrightarrow Q)$

Q Show that the statement $q \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$ is a Tautology.

P	q	$\neg p$	$\neg q$	$p \wedge \neg q$	$\neg p \wedge \neg q$	$q \vee (p \wedge \neg q)$	$q \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$
1	0	0	1	1	0	1	1
1	1	0	0	0	0	1	1
0	1	1	0	0	0	1	1
0	0	1	1	0	1	0	1

all the truth values of the compound statement is true. $\therefore$ It is a Tautology.

Q Construct a truth table for each of the following Compound Statement and determine which of the compound statements are tautologies

(a) $\neg (p \vee \neg q) \rightarrow \neg p$

(b) $p \rightarrow (q \rightarrow r)$

(c) $p \rightarrow (q \rightarrow p)$

(d) $[p \wedge (p \rightarrow q)] \rightarrow q$

(e) $q \leftrightarrow (\neg p \vee \neg q)$

(f) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow p \rightarrow r$

(a)

P	q	$\neg P$	$\neg q$	$P \vee \neg q$	$\neg(P \vee \neg q)$	$\neg(P \vee \neg q) \rightarrow \neg P$
1	0	0	1	1	0	1
1	1	0	0	1	0	1
0	1	1	0	0	1	1
0	0	1	1	1	0	1

$\therefore$ It is a Tautology

(b)

P	q	r	$q \rightarrow r$	$P \rightarrow (q \rightarrow r)$
1	1	1	1	1
1	1	0	0	0
1	0	1	1	1
1	0	0	1	1
0	1	1	1	1
0	1	0	0	1
0	0	1	1	1
0	0	0	1	1

$\therefore$ not a tautology.

(c)

P	q	$q \rightarrow P$	$P \rightarrow (q \rightarrow P)$
1	0	1	1
1	1	1	1
0	1	0	1
0	0	1	1

$\therefore$ It is a tautology

(d) $[P \wedge (P \rightarrow Q)] \rightarrow Q$

P	Q	$P \rightarrow Q$	$P \wedge (P \rightarrow Q)$	$P \wedge (P \rightarrow Q) \rightarrow Q$
1	0	0	0	1
0	1	1	0	1
1	1	1	1	1
0	0	1	0	1

all the truth values of statements are true.
 $\therefore$ It is a tautology.

(e) $Q \leftrightarrow (\neg P \vee \neg Q)$

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$Q \leftrightarrow \neg P \vee \neg Q$
1	0	0	1	1	0
0	1	1	0	1	1
1	1	0	0	0	0
0	0	1	1	1	1

not a tautology.

(f) $(P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow P \rightarrow R$

P	Q	R	$P \rightarrow Q$	$Q \rightarrow R$	$P \rightarrow R$	$(P \rightarrow Q) \wedge (Q \rightarrow R)$	$(P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow P \rightarrow R$
1	1	1	1	1	1	1	1
1	1	0	1	0	0	0	1
1	0	1	0	1	1	0	1
0	1	1	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	1	0	1
1	0	0	0	1	0	0	1
0	0	0	1	1	1	1	1

tautology

Logical Equivalence: The laws of Logic

Defn:- Two statements S_1, S_2 are said to be logically equivalent and we write $S_1 \Leftrightarrow S_2$ when the statement S_1 is true (false) iff the statement S_2 is (true (false)).

i.e., when $S_1 \Leftrightarrow S_2$ the statements S_1 and S_2 will have the same truth tables. OR $S_1 \Leftrightarrow S_2$ is a tautology.

Example:-

Q. S.T the statements $(P \rightarrow Q)$ and $(\neg P \vee Q)$ are logically equivalent
or $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

P	Q	$P \rightarrow Q$	$\neg P$	$(\neg P \vee Q)$	$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$
1	0	0	0	0	1
0	1	1	1	1	1
1	1	1	0	1	1
0	0	1	1	1	1

Here we can see that the truth values of $(P \rightarrow Q)$ and $(\neg P \vee Q)$ are same. or In other words, $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$ is a tautology.

8. S.T $P \leftrightarrow Q$ is logically equivalent to 100
 $(P \rightarrow Q) \wedge (Q \rightarrow P)$

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$P \leftrightarrow Q$	$P \rightarrow Q \wedge Q \rightarrow P$
1	0	0	1	0	0
0	1	1	0	0	0
1	1	1	1	1	1
0	0	1	1	1	1

column (5) & (6) are equal. $\therefore$ the statements
 $P \leftrightarrow Q$ & $(P \rightarrow Q) \wedge (Q \rightarrow P)$ are logically equivalent

9. S.T $P \wedge (\sim Q \vee R)$, $P \vee (Q \wedge \sim R)$ are not logically equivalent.

P	Q	R	$\sim Q$	$\sim R$	$\sim Q \vee R$	$P \wedge (\sim Q \vee R)$	$(Q \wedge \sim R)$	$P \vee (Q \wedge \sim R)$
1	1	1	0	0	1	1	0	1
1	1	0	0	1	0	0	1	1
1	0	1	1	0	1	1	0	1
0	1	1	0	0	1	0	0	0
0	0	1	1	0	1	0	0	0
0	1	0	0	1	0	0	1	0
1	0	0	1	1	1	1	0	1
0	0	0	1	1	1	0	0	0

clearly $P \wedge (\sim Q \vee R) \not\leftrightarrow P \vee (Q \wedge \sim R)$

The Laws of Logic

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For any primitive statements P, Q, R any tautology T_0 , and any contradiction F_0

1 $\neg \neg P \iff P$ Law of double negation

2 $\neg(P \vee Q) \iff \neg P \wedge \neg Q$
 $\neg(P \wedge Q) \iff \neg P \vee \neg Q$ } De Morgan's Law

3 $P \vee Q \iff Q \vee P$
 $P \wedge Q \iff Q \wedge P$ } Commutative laws.

4 $P \vee (Q \vee R) \iff (P \vee Q) \vee R$
 $P \wedge (Q \wedge R) \iff (P \wedge Q) \wedge R$ } Associative Laws.

5 $P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$
 $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$ } Distributive Laws

6 $P \vee P \iff P$
 $P \wedge P \iff P$ } Idempotent laws.

7 $P \vee F_0 \iff P$
 $P \wedge T_0 \iff P$ } Identity laws.

8 $P \vee \neg P \iff T_0$
 $P \wedge \neg P \iff F_0$ } Inverse Laws.

$$9. \left. \begin{aligned} P \vee T_0 &\Leftrightarrow T_0 \\ P \wedge F_0 &\Leftrightarrow F_0 \end{aligned} \right\} \text{ Domination Laws.}$$

$$10. \left. \begin{aligned} P \vee (P \wedge q) &\Leftrightarrow P \\ P \wedge (P \vee q) &\Leftrightarrow P \end{aligned} \right\} \text{ Absorption Laws.}$$

H.W

Problems:-

1- Let P, q, r denote primitive statements. Use truth tables to verify the following logical equivalences.

$$(i) \quad P \rightarrow (q \wedge r) \Leftrightarrow (P \rightarrow q) \wedge (P \rightarrow r)$$

$$(ii) \quad (P \vee q) \rightarrow r \Leftrightarrow (P \rightarrow r) \wedge (q \rightarrow r)$$

$$(iii) \quad P \rightarrow (q \vee r) \Leftrightarrow \neg r \rightarrow (P \rightarrow q)$$

2 Without using truth table, prove the following

$$(i) \quad (\neg P \vee q) \wedge (P \wedge (P \wedge q)) \equiv P \wedge q$$

$$(ii) \quad P \rightarrow (q \rightarrow P) \equiv \neg P \rightarrow (P \rightarrow q)$$

$$(iii) \quad \neg (P \leftrightarrow q) \equiv (P \vee q) \wedge \neg (P \vee q) \equiv (P \wedge \neg q) \vee (\neg P \wedge q)$$

(laws) and their duals which will be of use later are given in Tables 1.9, 1.10 and 1.11. They can be easily established by using truth tables.

Table 1.9 Laws of Algebra of Propositions

Sl. No.	Name of the law	Primal form	Dual form
1.	Idempotent law	$p \vee p \equiv p$	$p \wedge p \equiv p$
2.	Identity law	$p \vee F \equiv p$	$p \wedge T \equiv p$
3.	Dominant law	$p \vee T \equiv T$	$p \wedge F \equiv F$
4.	Complement law	$p \vee \neg p \equiv T$	$p \wedge \neg p \equiv F$
5.	Commutative law	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
6.	Associative law	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
7.	Distributive law	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
8.	Absorption law	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
9.	De Morgan's law	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$

Table 1.10 Equivalences Involving Conditionals

1.	$p \rightarrow q \equiv \neg p \vee q$
2.	$p \rightarrow q \equiv \neg q \rightarrow \neg p$
3.	$p \vee q \equiv \neg p \rightarrow q$
4.	$p \wedge q \equiv \neg(p \rightarrow \neg q)$
5.	$\neg(p \rightarrow q) \equiv p \wedge \neg q$
6.	$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
7.	$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
8.	$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
9.	$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Table 1.11 Equivalences Involving Biconditionals

1. $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
2. $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
3. $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
4. $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

TAUTOLOGICAL IMPLICATION

A compound proposition $A(p_1, p_2, \dots, p_n)$ is said to *tautologically imply* or simply *imply* the compound proposition $B(p_1, p_2, \dots, p_n)$, if B is true whenever A is true or equivalently if and only if $A \rightarrow B$ is a tautology. This is denoted by $A \Rightarrow B$, read as “ A implies B ”.

Note $\Rightarrow$ is not a connective and $A \Rightarrow B$ is not a proposition). For example, $p \Rightarrow p \vee q$, as seen from the following truth Table 1.12. We note that $p \vee q$ is true, whenever p is true and that $p \rightarrow (p \vee q)$ is a tautology.

Table 1.12

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Similarly we note that $(p \rightarrow q) \Rightarrow (\neg q \rightarrow \neg p)$ from the following truth Table 1.13.

Table 1.13

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$	$(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Some important implications which can be proved by truth tables are given in Table 1.14.

Table 1.14 Implications

1. $p \wedge q \Rightarrow p$
2. $p \wedge q \Rightarrow q$
3. $p \Rightarrow p \vee q$
4. $\neg p \Rightarrow p \rightarrow q$
5. $q \Rightarrow p \rightarrow q$
6. $\neg(p \rightarrow q) \Rightarrow p$
7. $\neg(p \rightarrow q) \Rightarrow \neg q$
8. $p \wedge (p \rightarrow q) \Rightarrow q$
9. $\neg q \wedge (p \rightarrow q) \Rightarrow \neg p$
10. $\neg p \wedge (p \vee q) \Rightarrow q$
11. $(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow p \rightarrow r$
12. $(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r) \Rightarrow r$

$$(i) (\neg P \vee Q) \wedge [P \wedge (P \wedge Q)] \equiv (\neg P \vee Q) \wedge [(P \wedge P) \wedge Q]$$

$$\equiv (\neg P \vee Q) \wedge (P \wedge Q)$$

by associative law.

idempotent law $P \wedge P = P$

$$\equiv (P \wedge Q) \wedge (\neg P \vee Q)$$

by commutative law.

$$\equiv [(P \wedge Q) \wedge \neg P] \vee [(P \wedge Q) \wedge Q]$$

by distributive law

$$\equiv [\neg P \wedge (P \wedge Q)] \vee [(P \wedge Q) \wedge Q]$$

by commutative law.

$$\equiv [(\neg P \wedge P) \wedge Q] \vee [P \wedge (Q \wedge Q)]$$

by associative law.

$$\equiv [F \wedge Q] \vee [P \wedge Q]$$

by Inverse & idempotent laws.

$$\equiv F \vee (P \wedge Q)$$

by Dominant law.

$$\equiv (P \wedge Q) \text{ by identity laws.}$$

we have

$$P \vee F_0 \Leftrightarrow P$$

$$P \wedge T_0 \Leftrightarrow P$$

$$(ii) P \rightarrow (Q \rightarrow P) \equiv \neg P \rightarrow (P \rightarrow Q)$$

$$\text{L.H.S} \Rightarrow P \rightarrow (Q \rightarrow P) \equiv \neg P \vee (Q \rightarrow P)$$

$$\equiv \neg P \vee (\neg Q \vee P)$$

$$\equiv \neg P \vee (P \vee \neg Q)$$

commutative law

Since conditional Statement

$$(P \rightarrow Q) \equiv (\neg P \vee Q)$$

similarly

$$(\neg Q \vee P \equiv Q \rightarrow P)$$

$$\equiv (\neg P \vee P) \vee \neg Q \quad \text{associative law}$$

$$\equiv (T_0 \vee \neg Q) \quad \text{inverse law.}$$

$$\equiv T_0 \quad \text{Dominant law.} \quad \text{--- (1)}$$

Now

$$\text{R.H.S} \equiv \neg P \rightarrow (P \rightarrow Q)$$

$$\equiv P \vee (P \rightarrow Q) \quad \text{law of double negation \& conditional statement}$$

$$\equiv P \vee (\neg P \vee Q) \quad \text{conditional logic}$$

$$\Rightarrow (P \vee \neg P) \vee Q \quad \text{associative law.}$$

$$\equiv T_0 \vee Q \quad \text{by inverse law.}$$

$$\equiv T_0 \quad \text{by dominant law.} \quad \text{--- (2)}$$

From (1) & (2) .. we can say

$$\underline{\underline{P \rightarrow (Q \rightarrow P) \equiv \neg P \rightarrow (P \rightarrow Q)}}$$

Equivalences Involving Biconditional

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$$1) P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$$

$$2) P \leftrightarrow Q \equiv \neg P \leftrightarrow \neg Q$$

$$3) P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$$

$$4) \neg(P \leftrightarrow Q) \equiv P \leftrightarrow \neg Q$$

Q Without using truth tables, prove the following

$$\neg(P \leftrightarrow Q) \equiv (P \vee Q) \wedge \neg(P \wedge Q) \equiv (P \wedge \neg Q) \vee (\neg P \wedge Q)$$

Proof:-

$$\neg(P \leftrightarrow Q) \equiv \neg((P \rightarrow Q) \wedge (Q \rightarrow P)) \quad \text{biconditional equivalence}$$

$$\equiv \neg((\neg P \vee Q) \wedge (\neg Q \vee P)) \quad \text{conditional equivalence}$$

$$\equiv \neg[(\neg P \vee Q) \wedge \neg Q] \vee [(\neg P \vee Q) \wedge P] \quad \text{distributive law}$$

$$\equiv \neg[(\neg P \wedge \neg Q) \vee (Q \wedge \neg Q)] \vee [(\neg P \wedge P) \vee (Q \wedge P)] \quad \text{by distributive law}$$

$$\equiv \neg[(\neg P \wedge \neg Q) \vee F] \vee [F \vee (Q \wedge P)] \quad \text{Inverse laws}$$

$$\equiv \neg[(\neg P \wedge \neg Q) \vee (Q \wedge P)] \quad \text{identity laws}$$

$$\equiv \neg(\neg(p \vee q) \vee (q \wedge p)) \text{ De Morgan's law } 108$$

$$\equiv (p \vee q) \wedge \neg(q \wedge p) \text{ De Morgan's law } \textcircled{1}$$

$$\equiv (p \vee q) \wedge (\neg q \vee \neg p) \text{ De Morgan's law}$$

$$\equiv [(p \vee q) \wedge \neg q] \vee [(p \vee q) \wedge \neg p] \text{ by distributive laws}$$

$$\equiv (p \wedge \neg q) \vee (q \wedge \neg q) \vee (p \wedge \neg p) \vee (q \wedge \neg p) \text{ by distributive laws.}$$

$$\equiv [(p \wedge \neg q) \vee F] \vee [\neg p \vee (q \wedge \neg p)] \text{ Inverse law.}$$

$$\equiv [(p \wedge \neg q) \vee (q \wedge \neg p)] \text{ identity laws } \textcircled{2}$$

$\equiv$ From $\textcircled{1}$ & $\textcircled{2}$; we can write,

$$\neg(p \leftrightarrow q) \equiv (p \vee q) \wedge \neg(p \wedge q) \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$$

Q) Without using truth tables, P.T the following

$$P \rightarrow (Q \rightarrow R) \iff (P \wedge Q) \rightarrow R$$

Proof: -

$$P \rightarrow (Q \rightarrow R) \iff \neg P \vee (Q \rightarrow R) \text{ Conditional equivalence}$$

$$\iff \neg P \vee (\neg Q \vee R) \text{ " "}$$

$$\iff (\neg P \vee \neg Q) \vee R \text{ associative law}$$

$$\iff \neg(P \wedge Q) \vee R \text{ De Morgan's law}$$

$$\iff (P \wedge Q) \rightarrow R \text{ Conditional equivalence}$$

Q. S.T. $[\neg P \wedge (\neg Q \wedge R)] \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$

Without using truth tables.

Proof:- L.H.S is,

$$\neg P \wedge (\neg Q \wedge R) \vee (Q \wedge R) \vee (P \wedge R)$$

$$\Leftrightarrow \neg P \wedge (\neg Q \wedge R) \vee [(Q \vee P) \wedge R] \text{ distributive law.}$$

$$\Leftrightarrow (\neg P \wedge \neg Q) \wedge R \vee [(Q \vee P) \wedge R] \text{ associative law}$$

$$\Leftrightarrow [\neg(P \vee Q) \wedge R] \vee [(Q \vee P) \wedge R] \text{ De Morgan's law.}$$

$$\Leftrightarrow [\neg(P \vee Q) \wedge R] \vee [(P \vee Q) \wedge R] \text{ commutative law.}$$

$$\Leftrightarrow [\neg(P \vee Q) \vee (P \vee Q)] \wedge R \text{ distributive law.}$$

$$\Leftrightarrow T_0 \wedge R$$

$$\left[\begin{array}{l} P \vee \neg P = T_0 \\ \therefore P \wedge \neg P = F_0 \\ \text{Inverse laws.} \end{array} \right]$$

$$\Leftrightarrow R$$

$$= \text{R.H.S}$$

$$P \wedge T_0 = P$$

identity law.

Defn:-

Let S be a statement. If S contains no logical connectives other than $\wedge$ and $\vee$, then the dual of S , denoted by S^d is the statement obtained from S by replacing each occurrence of $\wedge$ and $\vee$ by $\vee$ and $\wedge$ respectively and each occurrence of T_0 and F_0 by F_0 and T_0 respectively.

If p is any primitive statement; then p^d is the same as p .

$(\neg P)^d$ is same as $\neg P$

The statements $P \vee \neg P$ and $P \wedge \neg P$ are duals of each other whenever P is primitive and so are the statements

$P \vee T_0$ and $P \wedge F_0$

Exeg:

Let $S: (P \wedge \neg Q) \vee (R \wedge T_0)$

dual of S is, $S^d: (P \vee \neg Q) \wedge (R \vee F_0)$

Theorem:- The principle of Duality:-

Let S and t be statements that contains no logical connectives other than $\wedge$ and $\vee$
if $S \Leftrightarrow t$, then $S^d \Leftrightarrow t^d$

Substitution rules:-

- (1) Suppose that the compound statement P is a tautology. If ' p ' is a primitive statement that appears in P and we replace each occurrence of p by the same statement q , then the resulting compound statement P_1 is also a tautology.
- (2) Let P be a compound statement where ' p ' is an arbitrary statement that appears in P , and let q be a statement such that $q \Leftrightarrow p$. Suppose that in P we replace one or more occurrences of p by q . Then this replacement yields the compound statement P_1 . Under these circumstances
- $$P_1 \Leftrightarrow P.$$

Q. Use the substitution rule to verify that each of the following is a tautology

(a) $[p \vee (q \wedge r)] \vee \neg (p \vee (q \wedge r))$

Let S is the compound statement

$$S: p \vee (q \wedge r)$$

then we have,

$$S \vee \neg S \Leftrightarrow T_0$$

replace ~~$p \vee (q \wedge r)$~~ by $p \vee (q \wedge r)$ and the result follows, from 1st substitution rule

$$[p \vee (q \wedge r)] \vee \neg (p \vee (q \wedge r))$$

$$\Leftrightarrow S \vee \neg S$$

$$\Leftrightarrow \underline{\underline{T_0}}$$

(b) $(p \vee q) \rightarrow r \Leftrightarrow \neg r \rightarrow (p \vee q)$

For any primitive statements s, t we have: $(s \rightarrow t) \Leftrightarrow (\neg t \rightarrow \neg s)$

Replace each occurrence of s by $p \vee q$ and t by r , and the result is obtained by 1st substitution rule

⑧ Negate and simplify the compound statement

$$(p \vee q) \rightarrow r$$

We have,

$$(p \vee q) \rightarrow r \Leftrightarrow \neg(p \vee q) \vee r \quad (\text{logical equivalence})$$

Negating the statement

$$\neg[(p \vee q) \rightarrow r] \Leftrightarrow \neg(\neg(p \vee q) \vee r)$$

$$\Leftrightarrow (p \vee q) \wedge \neg r \quad \text{using De Morgan's law}$$

$$\therefore \underline{\underline{\neg[(p \vee q) \rightarrow r] \Leftrightarrow (p \vee q) \wedge \neg r}}$$

Q Negate the following sentences.

If Joan goes to Lake George, then Mary will pay for Joan's shopping spree.

Let p : Joan goes to Lake George.

q : Mary pays for Joan's shopping spree.

So we have $p \rightarrow q$. We want to find the negation of this.

$$P \rightarrow Q \Leftrightarrow \neg P \vee Q$$

$$\neg(P \rightarrow Q) \Leftrightarrow \neg(\neg P \vee Q) \\ \Rightarrow P \wedge \neg Q$$

$\therefore \neg(P \rightarrow Q)$: Joan goes to Lake George but Mary does not pay for Joan's shopping spree.

(8) Norma is doing her homework, and Karen is practising her piano lessons.

p : Norma is doing her homework

q : Karen is practising her piano lessons.

So we have $P \wedge Q$

$$\therefore \neg(P \wedge Q) \Rightarrow \neg P \vee \neg Q$$

i.e., Norma is not doing her homework

or Karen is not practising her piano

lessons.

(9) If Harold passes his C++ course and finishes his datastructures project, then he will graduate at the end of the semester

P : Harold passes his C++ course

Q : He finishes his data structure project

x : He will graduate at the end of the semester

$\therefore$ we have $p \wedge q \Rightarrow x$

Negation of this is,

$$\neg[(p \wedge q) \rightarrow x] \Leftrightarrow \neg[\neg(p \wedge q) \vee x]$$

$$\Leftrightarrow (p \wedge q) \wedge \neg x$$

$\therefore$ Harold did pass his C++ course and he did ~~not~~ finish his data structures project but he did not graduate at the end of the semester.

Q Negate the following

$$\neg[\neg[(p \vee q) \wedge r] \vee \neg q] \quad \text{where } p, q, r$$

are primitive statements.

$$\neg[\neg[(p \vee q) \wedge r] \vee \neg q] \Leftrightarrow \neg[\neg[(p \vee q) \wedge r] \wedge \neg \neg q]$$

$$\Leftrightarrow [(p \vee q) \wedge r] \wedge q \quad \begin{matrix} \text{(De Morgan's Law)} \\ \text{(Law of double negation)} \end{matrix}$$

$$\Leftrightarrow (p \vee q) \wedge (r \wedge q) \quad \text{associative Law}$$

$$\Leftrightarrow (p \vee q) \wedge (q \wedge r) \quad \text{Commutative Law}$$

$$\iff [(p \vee q) \wedge q] \wedge r$$

associative law

$$\iff q \wedge r$$

Since $p \wedge (p \vee q) \Rightarrow p$.
Absorption law.
here $q \wedge (p \vee q) \Rightarrow q$

$$\therefore \underline{\underline{\neg [\neg [(p \vee q) \wedge r] \vee \neg q] \iff q \wedge r}}$$

Q For primitive statements P, Q , Does there exist any simpler way to express the compound statement $(p \vee q) \wedge \neg (\neg p \wedge q)$ which is logically equivalent to the given.

Consider

$$(p \vee q) \wedge \neg (\neg p \wedge q)$$

$$\begin{aligned} \because \neg (\neg p \wedge q) \\ &= \neg \neg p \vee \neg q \\ &= p \vee \neg q \end{aligned}$$

$$\iff (p \vee q) \wedge (p \vee \neg q) \quad \begin{array}{l} \text{Demorgan's law} \\ \text{and} \\ \text{law of double negation} \end{array}$$

$$\iff p \vee (q \wedge \neg q) \quad \begin{array}{l} \text{distributive law of } \vee \\ \text{over } \wedge \end{array}$$

$$\iff p \vee F_0 \quad \text{Inverse law.}$$

$$\iff p \quad \text{identity law.}$$

$\therefore (p \vee q) \wedge \neg (\neg p \wedge q) \iff p$ So we can express the given compound statement by a simple equivalent statement P .

Converse, Inverse & Contrapositive Sentence

If $P \rightarrow Q$ is a statement then

➤ $Q \rightarrow P$ is the converse

➤ $\neg P \rightarrow \neg Q$ is the inverse

➤ $\neg Q \rightarrow \neg P$ is the contrapositive of the statement $P \rightarrow Q$

Ex: If it is a dog then it is an animal

$$P \rightarrow Q$$

* Converse of this is $Q \rightarrow P$

ie, If it is an animal, then it is a dog

* Inverse of this is, $\neg P \rightarrow \neg Q$

If it is not a dog then it is not an animal

* Contrapositive of this is, $\neg Q \rightarrow \neg P$

If it is not an animal, then it is not a dog.

Logical Implication: Rules of Inference.Defn:- Argument:-

An argument is a sequence of statements called premises which end with a conclusion.

Consider the general form of an argument

and suppose $(P_1 \wedge P_2 \wedge \dots \wedge P_n) \rightarrow Q$

where $P_1, P_2, \dots, P_n$ are called premises of the argument and 'Q' is called the conclusion of the argument.

If $(P_1 \wedge P_2 \wedge \dots \wedge P_n) \rightarrow Q$ is a tautology then the argument is termed as valid.

Other wise termed as Invalid.

We can check the validity of a given statement either by Truth table Technique or by using Rules of Inference.

Type: 1 Truth table Technique:-

Suppose we have a conclusion 'C' follows from a set of premises $H_1, H_2, \dots, H_n$ then we can check the validity of the

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argument by using the truth values of argument

The argument $H_1 \wedge H_2 \wedge \dots \wedge H_n \rightarrow C$ is valid if $H_1 \wedge H_2 \wedge \dots \wedge H_n \rightarrow C$ is a tautology. Otherwise we can say the argument is not valid.

Q. Using a Truth table check the validity of the statements:

If there is cream, then I will drink coffee

If there is a donut, then I will drink coffee

There is no cream and there is a donut

Therefore I drink coffee.

Ans:- Let us define the primitive statements

P: There is cream.

Q: I will drink coffee

R: There is a donut

So we have the following premises

1) $P \rightarrow Q$

2) $R \rightarrow Q$

3) $\sim P \wedge R$

and from these (1) (2) (3) premises we have a conclusion 2

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$$\begin{array}{l} \text{ie, } p \rightarrow q \\ r \rightarrow q \\ \hline \sim p \wedge q \end{array}$$

$\therefore q$

So we need to s.t $(p \rightarrow q) \wedge (r \rightarrow q) \wedge (\sim p \wedge q) \rightarrow q$ is a tautology.

ie, $H_1 \wedge H_2 \wedge H_3 \rightarrow C$ is a tautology

			★	★	★			
P	q	r	$P \rightarrow q$	$r \rightarrow q$	$\neg P$	$\neg P \wedge q$	$H_1 \wedge H_2 \wedge H_3$	$H_1 \wedge H_2 \wedge H_3 \rightarrow C$
1	0	0	0	1	0	0	0	1
0	0	1	1	0	1	1	0	1
0	1	0	1	1	1	0	0	1
0	1	1	1	1	1	1	1	1
1	0	1	0	0	0	0	0	1
1	1	0	1	1	0	0	0	1
0	0	0	1	1	1	0	0	1
1	1	1	1	1	0	0	0	1

clearly $H_1 \wedge H_2 \wedge H_3 \rightarrow C$ is a tautology so the given statement is valid.

Theory of Inference

Inference theory is concerned with the inferring of a conclusion from certain hypotheses or basic assumptions, called premises by applying certain principles of reasoning called rules of inference.

The rules of inference are used to obtain a conclusion from a set of premises in a finite sequence of steps, called argument.

Any conclusion which is arrived at ^{by following} these rules is called a valid conclusion and the argument is called a valid argument.

Rules of Inference

Rule P:— A premise may be introduced at any step in the derivation (given)

Rule T:— A formula ϕ may be introduced in the derivation

Rules of Inference

1. Modus Ponens

$$P, P \rightarrow Q \Rightarrow Q$$

2. Modus Tollens

$$\neg Q, P \rightarrow Q \Rightarrow \neg P$$

3. Disjunctive syllogism.

$$\neg P, P \vee Q \Rightarrow Q$$

$$\neg Q, P \vee Q \Rightarrow P$$

4. Simplification

$$P \wedge Q \Rightarrow P, Q$$

5. Conjunction

$$P, Q \Rightarrow P \wedge Q$$

6. Hypothetical syllogism:-

$$P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$$

7. Dilemma:-

$$P \vee Q, P \rightarrow R, Q \rightarrow R \Rightarrow R$$

along with these rules, sometime we may have to use conditional equivalence, Demorgan's law, compliment laws etc.

Problems:-

1. Prove that 'S' is a valid conclusion from the premises $p \rightarrow \sim q$, $q \vee r$, $\sim S \rightarrow R$ and $\sim r$.

Step	Statement	Rule	Justification
1.	$\sim r$	P	_____
2.	$q \vee r$	P	_____
3.	q	T	(1) & (2) disjunctive syllogism.
4.	$p \rightarrow \sim q$	P	_____
5.	$\sim p$	T	(3) & (4) Modus tollens.
6.	$\sim S \rightarrow P$	P	_____
7.	$\sim(\sim S)$	T	(5) & (6) Modus tollens
8.	S	T	double negation.

S is a valid conclusion

- ② Prove that $r \wedge (p \vee q)$ is a valid conclusion from the premises $p \vee q, q \rightarrow r, p \rightarrow m, \sim m$

Proof:-

step	statement	Rule	Justification
1.	$\sim m$	P	—
2.	$p \rightarrow m$	P	—
3.	$\sim p$	T	① & ② Modus tollens
4.	$p \vee q$	P	—
5.	q	T	③ & ④ disjunctive syllogism
6.	$q \rightarrow r$	P	—
7.	r	T	⑤ & ⑥ Modus ponens
8.	$r \wedge (p \vee q)$	T	④ & ⑦ conjunction

$\therefore r \wedge (p \vee q)$ is a valid conclusion

Q. S.T. $t \wedge s$ can be derived from the premises $p \rightarrow q$, $q \rightarrow \neg r$, r , $p \vee (t \wedge s)$

Step	Statement	Rule	Justification.
1.	r	P	—
2.	$q \rightarrow \neg r$	P	—
3.	$\neg q$	T	(1) & (2) Modus tollens
4.	$p \rightarrow q$	P	—
5.	$\neg p$	T	(3) & (4) Modus tollens
6.	$p \vee (t \wedge s)$	P	—
7.	$t \wedge s$	T	(5) & (6) disjunctive syllogism.

Hence proved

Q. Show that $a \vee b$ follows from the premises $p \vee q$, $p \vee q \rightarrow \neg r$, $\neg r \rightarrow s \wedge t$
 $s \wedge t \rightarrow a \vee b$

step	statement	Rule	Justification
1.	$p \vee q$	P	—
2.	$(p \vee q) \rightarrow \neg r$	P	—
3.	$\neg r$	T	① & ② Modus ponens
4.	$\neg r \rightarrow s \wedge \neg t$	P	—
5.	$s \wedge \neg t$	T	③ & ④ Modus ponens
6.	$s \wedge \neg t \rightarrow a \vee b$	P	—
7.	$a \vee b$	T	⑤ & ⑥ Modus ponens

Hence proved

Q. Prove the validity of the following argument
 "If I get the job and work hard, then I will get promoted.
 If I get promoted then I will be happy.
 I will not be happy. Therefore
 either I will not get the job or I will not work hard."

Let p : I get the job

q : I work hard

r : I get promoted

s : I will be happy

Then the above arguments can be written symbolically as

$$p \wedge q \Rightarrow r$$

$$r \rightarrow s$$

$$\neg s$$

$\therefore \neg p \vee \neg q$ is conclusion

ie, we have to S.T from these premises
we can get the conclusion as $\neg p \vee \neg q$

Step	Statement	Rule	Justification
1.	$\neg s$	P	—
2.	$r \rightarrow s$	P	—
3.	$\neg r$	T	(1) & (2) Modus tollens
4.	$p \wedge q \rightarrow r$	P	—
5.	$p \wedge q \rightarrow s$	T	(4) & (2) hypothetical syllogism
6.	$\neg(p \wedge q) = \neg p \vee \neg q$	T	(1) & (5) Modus tollens

$\therefore \neg p \vee \neg q$ is a valid conclusion.

Q If there are meeting, then travelling was difficult.

If they arrived on time, then travelling was not difficult.

They arrived on time.

There was no meeting

Show that these statements constitute a valid argument

Proof:-

Let p = there is a meeting

q = travelling was difficult.

r = they arrived on time

Then we have following arguments.

$$p \rightarrow q$$

$$r \rightarrow \neg q$$

$$r$$

$\therefore \neg p$ is the conclusion.

We have to prove that, $\neg p$ is a valid conclusion.

Step	Statement	rule	Justification
------	-----------	------	---------------

- | | | | |
|----|---------------------------|----------|---------------|
| 1. | $\neg$ | P | _____ |
| 2. | $\neg \rightarrow \neg q$ | $\neg P$ | _____ |
| 3. | $\neg q$ | T | ① & ② Modus p |
| 4. | $P \rightarrow q$ | P | _____ |
| 5. | $\neg p$ | T | ③ & ④ Modus |

$\therefore$ Given: Premises forms a valid conclusion

Section 2.4 Quantifiers

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Defn: A part of a declarative sentence describing the properties of an object or relation among objects is called a predicate.

For exa: Consider 2 preposition

Ram is a bachelor

Shyam is a bachelor

Both Ram and Shyam has the same property of being bachelor.

The 2 prepositions can be replaced by a single statement 'x is a bachelor'

It can be symbolised by $B(x)$; x is a bachelor

Ram is a bachelor
Subject (x) predicate

Defn: - The domain or universal of discourse is the set of all possible values of 'x' that may be substituted in place of variables.

Quantifiers

A predicate becomes a proposition when we assign it fixed values. However another way to make a predicate into a proposition is to quantify it.

i.e., the predicate is true (or false)

for all possible values in the universe of discourse OR ^{for} some values(s) in the universe of discourse

Such quantification can be done with

2 quantifiers (1) The universal quantifier

(2) Existential quantifier

(1) The Universal quantifier:-

- Some statement telling about all individuals or objects belonging to a certain sets.
- These statements are beginning with 'for all', 'for every', 'everything', 'everyone' like that
- denoted as ' $\forall$ ' (for every)
- $\forall x P(x) \Rightarrow$ for all x , $P(x)$
- A proposition preceded by 'for every' always have a truth value.

Exa: 1 $\forall n \in \mathbb{N}, n+4 > 3$

The truth value of this statement is 1. (True)

Exa: 2 "All men are mortal"

This can be read as $\forall x$, if x is a man then x is mortal.

Let $M(x)$: x is a man

$H(x)$: x is mortal

The above statement we can represent as, $\forall x [M(x) \rightarrow H(x)]$

Exa: 3

Every apple is red

We can write this as,

$\forall x$, if x is an apple then x is red.

$\therefore$ Let $A(x)$: x is an apple

$R(x)$: x is ⁱⁿ red colour.

$\therefore$ We have,

$$\forall x \{A(x) \rightarrow R(x)\}$$

Exa: 4 Any integer is either +ve or -ve

$\forall x$, if x is an integer then x is either +ve or -ve

$$\forall x \quad I(x) \rightarrow (P(x) \vee N(x))$$

$I(x)$: x is an integer

$P(x)$: x is +ve

$N(x)$: x is -ve.

(2) Existential Quantifier

- These statements begin with "for some", "there exists" or "there is at least one"
- denoted as $\exists(x)$

Exa: 1 There exist a man.

$$\exists(x) M(x)$$

Exa: 2 Some men are clever.

This we can rewrite as, there exist x such that x is a man and x is clever.

$$\exists(x) [M(x) \wedge C(x)]$$

Where $M(x)$: x is a man

$C(x)$: x is clever.

Exa: 3 Some real numbers are rational

Let $R(x)$: x is a real no

$P(x)$: x is rational number

$$\exists(x) [R(x) \wedge P(x)]$$

(A) Rewrite the following argument using quantifiers variables and predicate symbols.

- (a) All birds can fly
- (b) Not all birds can fly
- (c) Some men are genius
- (d) Some numbers are not rational
- (e) There is a student who likes Mathematics but not geography.

Answers:-

(a) Let $B(x)$: x is a bird
 $F(x)$: x can fly

"All birds can fly" we can rewrite as For every x , if x is a bird, then it can fly

ie, $\forall x (B(x) \rightarrow F(x))$

(b) "Not all birds can fly" is the negation statement of all birds can fly.

$\therefore \neg \forall x [B(x) \rightarrow F(x)]$

(c) Some men are genius

Let $M(x)$: x is a man
 $G(x)$: x is genius

There exist x such that x is a man and genius.

$$\exists x [M(x) \wedge G(x)]$$

(d) Some numbers are not rational

Let $N(x)$: x is a number

$R(x)$: x is rational

There exist x such that x is a number but x is not rational

$$\exists x [N(x) \wedge \neg R(x)]$$

(e) There is a student who likes Maths but not geography.

Let $S(x)$: x is a student

$M(x)$: x likes Maths

$G(x)$: x likes geography.

$$\exists x [S(x) \wedge M(x) \wedge \neg G(x)]$$

8. All birds have two legs & two wings.

Let $B(x)$: x is a bird

$L(x)$: x have 2 legs

$W(x)$: x have 2 wings.

For every x , if x is a bird then it has 2 legs and 2 wings.

$$\forall x [B(x) \rightarrow L(x) \wedge W(x)]$$

9. Every child will receive either a gold fish or a beta fish.

$C(x)$: x is a child

$G(x)$: x receives gold fish

$B(x)$: x receives beta fish

$\therefore$ For every x , if x is a child then x will receive either a gold fish or a beta fish.

$$\forall x [C(x) \rightarrow G(x) \vee B(x)]$$

Negation of the quantified statement

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- 1) $\neg (\forall x P(x)) = \exists x \neg P(x)$
- 2) $\neg (\exists x P(x)) = \forall x \neg P(x)$
- 3) $\neg \forall x \neg P(x) = \exists x P(x)$
- 4) $\neg \exists x \neg P(x) = \forall x P(x)$

Negate the sentences:-

(i) All integers are greater than 8.

Let $I(x)$: x is an integer

$G(x)$: $x > 8$

$$\forall x [I(x) \rightarrow G(x)]$$

Now this is logically equivalent

$$\forall x [I(x) \rightarrow G(x)] \quad \left[P \rightarrow Q \Leftrightarrow \neg P \vee Q \right]$$

$$\therefore \neg \forall x [I(x) \rightarrow G(x)] = \exists x I(x) \wedge \neg G(x)$$

There exist an integer x such that x is ~~both~~ an integer but x is not greater than 8.

② For all real no. x , if $x > 3$ then $x^2 > 9$

$P(x)$: x is a real number > 3 .

$Q(x)$: $x^2 > 9$.

$\forall x [P(x) \rightarrow Q(x)]$. Negation of this is,

$$\neg \forall x [P(x) \rightarrow Q(x)] \Leftrightarrow \neg \forall x [\neg P(x) \vee Q(x)]$$

$$\Leftrightarrow \exists x P(x) \wedge \neg Q(x)$$

There exists a real no. x such that $x > 3$ but x^2 is not greater than 9.

③ Negate the following

$$\textcircled{a} \neg [\forall x P(x) \vee \exists y Q(y)] = \neg [\forall x (P(x) \vee \exists y Q(y))]$$

$$= \neg \forall x \neg P(x) \vee \exists y \neg Q(y)$$

$$\underline{\underline{= \exists x P(x) \wedge \forall y \neg Q(y)}}$$

Since $\neg(P \vee Q) = \neg P \wedge \neg Q$.

$$\textcircled{b} \neg \forall x P(x) \wedge \forall y Q(y)$$

$$\Leftrightarrow \neg \forall x P(x) \vee \neg \forall y Q(y)$$

$$\Leftrightarrow \exists x \neg P(x) \vee \exists y \neg Q(y)$$

$$\underline{\underline{\quad}}$$

$$\textcircled{c} \neg [\exists x p(x) \vee \forall y q(y)]$$

$$\iff \neg \exists x p(x) \wedge \neg \forall y q(y)$$

$$\iff \forall x \neg p(x) \wedge \exists y \neg q(y)$$

Problems: -

1. Let $p(x)$, $q(x)$ denote the following open statements.

$$p(x): x \leq 3$$

$$q(x): x+1 \text{ is odd}$$

If the universe consists of all integers, are the truth values of the following statements.

$$(a) q(1) \quad (b) \neg p(3) \quad (c) p(7) \vee q(7)$$

$$(d) p(3) \wedge q(4) \quad (e) \neg (p(-4) \vee q(-3))$$

$$(f) \neg p(-4) \wedge \neg q(-3)$$

2. For the universe of all integers let $p(x)$, $r(x)$, $s(x)$ and $t(x)$ be the following open statements

$$p(x): x > 0$$

$$q(x): x \text{ is even}$$

$$r(x): x \text{ is a perfect square}$$

$$s(x): x \text{ is exactly divisible by 4}$$

$$t(x): x \text{ is exactly divisible by 5.}$$

write the following in symbolic form and determine whether each of the $\forall x$ statements in part (a) is true or false. For each false statement provide a counter example

- (i) at least one integer is even
- (ii) There exists a positive integer that is even
- (iii) If x is even then x is not divisible by 5
- (iv) No even integer is divisible by 5
- (v) There exists an even integer divisible by 5
- (vi) If x is even and x is a perfect square, then x is divisible by 4.

Answers:-

$$P(x): x \leq 3$$

$$Q(x): x+1 \text{ is odd.}$$

1. (a) $Q(1) = 2 \text{ is odd}$ [false $\rightarrow 0$]. Truth value.

$$\neg P(x) = x > 3$$

(b) $\neg P(3) = 3 > 3$. false statement.

(c) $P(7) \vee Q(7) \Rightarrow 7 \leq 3 \text{ or } 7+1 \text{ is odd}$

$$\Rightarrow \text{false or false.}$$

$$\Rightarrow 0 \vee 0$$

$$\Rightarrow \underline{0.} \text{ (false)}$$

$$(d) P(3) : 3 \leq 3 \quad \text{True.}$$

$$Q(4) : 5 \text{ is odd} \quad \text{True.}$$

$$P(3) \wedge Q(4) = \underline{\underline{1 \wedge 1}} = 1 \quad (\text{True})$$

$$(e) \neg (P(-4) \vee Q(-3)) \Rightarrow \neg P(-4) \wedge \neg Q(-3)$$

$$P(-4) = -4 \leq 3 \quad (\text{True})$$

$$Q(-3) = -3 + 1 \text{ mod } 2 \quad (\text{false})$$

$$\neg P(-4) = \text{false.}$$

$$\neg Q(-3) = \text{True.}$$

$$\therefore \neg P(-4) \wedge \neg Q(-3) \Rightarrow \text{True} \wedge \text{false} \\ \Rightarrow \underline{\underline{\text{false}}}$$

$$(2) (i) \exists x. Q(x)$$

$$(ii) \exists x [P(x) \wedge Q(x)]$$

$$(iii) \forall x [Q(x) \rightarrow \neg t(x)]$$

$$(iv) \neg \forall x [Q(x) \rightarrow t(x)] \Rightarrow \neg \forall x [\neg Q(x) \vee t(x)] \\ \Rightarrow \exists x Q(x) \wedge \neg t(x)$$

$$(v) \forall x [Q(x) \rightarrow \neg t(x)]$$

$$(vi) \exists x Q(x) \wedge t(x)$$

$$(vii) \forall x [Q(x) \wedge r(x)] \rightarrow S(x)$$

Multiple Quantifiers

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We can consider the predicates which are functions of more than one variable. It is also possible to combine universal and existential quantifiers within a single statement involving more than one variable. In these cases, order of quantifiers are very important.

Exa: Consider the statement

$P(x, y)$: x admires y . Then

(1) Everyone admires everybody:

$$\forall x \forall y P(x, y)$$

(2) Everyone admires some one:

$$\forall x \exists y P(x, y)$$

(3) Some one admires everyone

$$\exists x \forall y P(x, y)$$

(4) Someone admires somebody

$$\exists x \exists y P(x, y)$$

(5) Everybody is admired by everyone
 $\forall y \forall x P(x, y)$

(6) Everybody is admired by someone
 $\forall y \exists x P(x, y)$

(7) There is some one whom everyone admires
 $\exists y \forall x P(x, y)$

(8) There is some one whom some one admires
 $\exists y \exists x P(x, y)$

Here (1) & (5) are logically equivalent.

(6) & (8) are logically equivalent

i.e., $\forall x \forall y P(x, y) \Leftrightarrow \forall y \forall x P(x, y)$

$\exists x \exists y P(x, y) \Leftrightarrow \exists y \exists x P(x, y)$

Defn:- Free & Bound. variables.

A formula containing a part of the
 $\forall x P(x)$ OR $\exists x P(x)$ such a part

called ~~an~~ x -bound part of the formula

Any variable appearing in an x -bound part of the formula is called bound variable. Otherwise it is called free variable. Any formula immediately following $\forall x$ or $\exists x$ is called scope of the quantifier.

Ex:-

$$\forall x P(x, y)$$

Here x is the bound variable
 y is the free variable, $P(x, y)$ is the scope of the quantifier.

$$\exists x [P(x) \wedge Q(x)]$$

x is bound variable

$P(x) \wedge Q(x)$ is the scope.

Predicate Calculus Rules:-

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(1) Rule US [universal specification]

$$\forall x P(x) \Rightarrow P(a)$$

(2) Rule ES [Existential specification]

$$\exists x P(x) \Rightarrow P(a)$$

removing
quantifier

• [Modus ponens, Modus tollens, disjunctive syllogism also we have to apply.]

(3) $\neg \forall x P(x) \Rightarrow \exists x \neg P(x)$

(4) $\neg (\exists x P(x)) \Rightarrow \forall x \neg P(x)$

(5) Rule UG [universal generalisation]

$$P(a) \Rightarrow \forall x P(x)$$

(6) Rule EG [Existential generalisation]

$$P(a) \Rightarrow \exists x P(x)$$

(5) & (6) are known as ~~adding~~ adding quantifier

⑧ P.T $\forall x [P(x) \rightarrow Q(x)], \exists x P(x)$
 $\Rightarrow \exists x Q(x)$

<u>Step</u>	<u>Statement</u>	<u>Rule</u>	<u>Justification</u>
1.	$\exists x P(x)$ removing quantifier	P	_____
2.	$P(a)$	E.S	from ①
3.	$\forall x [P(x) \rightarrow Q(x)]$ removing quantifier	P	_____
4.	$P(a) \rightarrow Q(a)$	U.S	step 3.
5.	$Q(a)$	T	② & ④ Modus ponens
6.	$\exists x Q(x)$	E.G	from ⑤.

(8.2) prove that $\forall x [P(x) \rightarrow Q(y) \wedge R(x)]$, $\exists x P(x)$ (147)
 $\Rightarrow Q(y) \wedge \exists x [P(x) \wedge R(x)]$

Step	Statement	Rule	Justification
		given Premises	—
1.	$\exists x P(x)$	E.S	Step ①.
2.	$P(a)$		
3.	$\forall x [P(x) \rightarrow Q(y) \wedge R(x)]$	P	—
4.	$P(a) \rightarrow Q(y) \wedge R(a)$	U.S	from ③.
5.	$Q(y) \wedge R(a)$	T	② & ④ Modus ponens.
6.	$Q(y)$	T	Simplification
7.	$R(a)$	T	Simplification
8.	$P(a) \wedge R(a)$	T.	Conjunction ② & ⑦.
9.	$\exists x [P(x) \wedge R(x)]$	E.G	Step 8.
10.	$Q(y) \wedge \exists x [P(x) \wedge R(x)]$	T	Conjunction ⑥ & ⑨.

⑧ P.T $\forall x [P(x) \rightarrow Q(x)], \exists x P(x) \Rightarrow \exists x Q(x)$

Step	Statement	Rule	Justification
1.	$\exists x P(x)$ Removing quantifier	P	_____
2.	$P(a)$	E.S	from (1)
3.	$\forall x [P(x) \rightarrow Q(x)]$ removing quantifier	P	_____
4.	$P(a) \rightarrow Q(a)$	U.S	step 3.
5.	$Q(a)$	T	(2) & (4) Modus ponens
6.	$\exists x Q(x)$	E.G	from (5).

⑨ S.T $\exists y \forall x P(x,y) \Rightarrow \forall x \exists y P(x,y)$ (Bridgman's Law)

Step	Statement	Rule	Justification
1.	$\exists y \forall x P(x,y)$	P	_____
2.	$\forall x P(x,a)$	E.S	from (1)
3.	$P(b,a)$	U.S	from (2)
4.	$\exists y P(b,y)$	E.G	from (3)
5.	$\forall x \exists y P(x,y)$	U.G	from (4)

(8) Test the validity of the argument

- (i) Everyone who knows Java will get a high paying job
- (ii) Someone in this class knows Java
- (iii) Therefore someone in this class will get a high paying job.

Let $P(x)$: x knows Java

$Q(x)$: x will get a high paying job

So we have, 2 premises.

(i) $\forall x$, if x is a person who knows Java then he will get a high paying job.

$$\text{i.e., } \forall x [P(x) \rightarrow Q(x)]$$

(ii) There exist a person x , who knows Java

$$\text{i.e., } \exists x P(x)$$

(iii) Conclusion is,

There exist a person x in this class who will get a high paying job

$$\exists x Q(x)$$

In the previous problem (1), we have already proved this

<u>Slip</u>	<u>statement</u>	<u>Rule</u>	<u>Justification</u>
1.	$\exists x P(x)$	P	_____
2.	$P(a)$	E-S	from (1)
3.	$\forall x P(x) \rightarrow q(x)$	P	_____
4.	$P(a) \rightarrow q(a)$	U-S	from (3)
5.	$q(a)$	T	(2) & (4) Modus ponens
6.	$\exists x q(x)$	E-G	from (5)

Note:-

Suppose the same problem is asked like this

"Every one who knows Java will get a high paying job"

"Ram knows Java"

Therefore Ram will get a high paying job"

Ans:- So we have, $\forall x P(x) \rightarrow q(x)$,

$$P(b) \Rightarrow q(b)$$

Let $P(x)$: x knows Java

$q(x)$: x will get a high paying job

b : Ram

$P(b)$ = Ram knows Java.

Step	Statement	Rule	Justification
1.	$\forall x P(x) \rightarrow q(x)$	P	_____
2.	$P(a) \rightarrow q(a)$	E.s	Step (1)
3.	$P(b)$	P	_____
4.	$q(b)$	T	(2) & (3) Modus ponens.

- ⑧ check the validity of the argument
- "All mammals are animals.
- Some mammals are two legged
- Therefore some animals are 2 legged"

Ans:- Let $M(x)$: x is a mammal

$A(x)$: x is an animal

$L(x)$: x have 2 legs.

Given, $\forall x [M(x) \rightarrow A(x)], \exists x [M(x) \wedge L(x)]$
 $\Rightarrow \exists x [A(x) \wedge L(x)]$

Step	Statement	Rule	Justification
1.	$\forall x [M(x) \rightarrow A(x)]$	P	_____
2.	$M(a) \rightarrow A(a)$	U.S	Step (1)
3.	$\exists x M(x) \wedge L(x)$	P	_____
4.	$M(a) \wedge L(a)$	E.S	Step (3)
5.	$M(a), L(a)$	T	Simplification (4)
6.	$A(a)$	T	(2) & (5) Modus ponens
7.	$A(a) \wedge L(a)$	T	(5) & (6) Conjunction
8.	<u>$\exists x A(x) \wedge L(x)$</u>	E.G	Step (7)

Q. If One person is more successful than another then he has worked harder to deserve Success.

John has not worked harder than peter.
 Therefore John is not successful than peter.

Let $S(x,y)$: x is more successful than
 $W(x,y)$: x has worked harder than
to deserve success.

$$\left. \begin{array}{l} \forall x \forall y [S(x,y) \rightarrow W(x,y)] \\ \neg W(a,b) \end{array} \right\} \begin{array}{l} a: \text{John} \\ b: \text{Pety} \end{array}$$

$$\Rightarrow \neg S(a,b)$$

<u>Step</u>	<u>Statement</u>	<u>Rule</u>	<u>Justification.</u>
1.	$\neg W(a,b)$	P	_____
2.	$\forall x \forall y [S(x,y) \rightarrow W(x,y)]$	P	_____
3.	$\forall y [S(a,y) \rightarrow W(a,y)]$	U. S	Step 2
4.	$S(a,b) \rightarrow W(a,b)$	U. S	Step 3.
5.	$\neg S(a,b)$	T	(1), (4) Modus tollen's.

So the conclusion $\forall x [p(x) \rightarrow \neg q(x)]$ follows from the premises $\exists x [p(x) \wedge q(x)] \rightarrow \forall y [r(y) \rightarrow s(y)]$ and $\exists y [r(y) \wedge \neg s(y)]$

Step	Statement	Rule	Justification.
1.	$\exists y [r(y) \wedge \neg s(y)]$	P	—
2.	$r(a) \wedge \neg s(a)$	E.S	step 1
3.	$\neg r(a) \rightarrow s(a)$	T	★ conditional equivalence $p \wedge q \Leftrightarrow \neg(p \rightarrow \neg q)$
4.	$\exists y [\neg(r(y) \rightarrow s(y))]$	E.G	step 3
5.	$\neg \forall y [r(y) \rightarrow s(y)]$	T	step (4) negation equivalence
6.	$\exists x [p(x) \wedge q(x)] \rightarrow \forall y [r(y) \rightarrow s(y)]$	P	—
7.	$\neg \exists x [p(x) \wedge q(x)]$	T	(5) & (6) Modus ponens
8.	$\forall x \neg (p(x) \wedge q(x))$	T	(7) negation equivalence
9.	$\neg (p(b) \wedge q(b))$	U.S	step 8.
10.	$\neg p(b) \vee \neg q(b)$	T	D'Morgan's law step 9
11.	$p(b) \rightarrow \neg q(b)$	T	★ conditional equivalence $\neg p \vee \neg q \Leftrightarrow p \rightarrow \neg q$
12.	$\forall x p(x) \rightarrow \neg q(x)$	U.G	step 11

Set Theory: -

Introduction: -

A set is a well-defined collection of objects called elements which share ^{some} common property.

Any object belongs to a set is called a member or an element of the set.

Sets will be denoted by capital letters ($A, B, C, \dots$) and elements will be denoted by small letters ($a, b, c, \dots$) of English Alphabet.

If x is an element of a set we write $x \in A$.

Subsets: -

Let A and B be two sets. If every elt of A is an elt of B , then A is called subset of B . Symbolically we write $A \subseteq B$.

If A is a subset of B and $A \neq B$ then we say A is a proper subset of B and we write $A \subset B$.

Equal sets: -

Two sets A and B are equal iff $A \subseteq B$ and $B \subseteq A$ & we write $A = B$.

Types of set

1. Null set

A set containing no element is called a null set denoted by ϕ or $\{ \}$

2. Singleton set

A set containing only a single element is called a singleton set.

3. Finite set

A set is said to be finite set if it consists of only finite number of elements

4. Infinite set:- A set is said to be an infinite set if it consists of infinite no. of elements.
5. Equal sets:- Two sets A and B are said to be equal if they contain same elements.
6. powerset:- Let A be any set. Then the collection of all subsets of A is called the power set of A and is denoted by $P(A)$.
7. Universal set:- In any discussion in set theory if all the sets under consideration are subsets of a fixed set U, then the set U is called the universal set.

Operation on sets:-

1. Union of elements

The union of two sets A and B is the set of all those elements which belongs to A or B or both. We use the notation $A \cup B$ to denote the union of A and B. i.e., $A \cup B = \{x: x \in A \text{ or } x \in B\}$

2. Intersection of elements

The intersection of two sets A and B denoted by $A \cap B$ is set consisting of all elements which belongs to both A and B

$$\text{i.e., } A \cap B = \{x: x \in A \text{ and } x \in B\}$$

3. Disjoint sets.

Two sets A and B are said to be disjoint if $A \cap B = \phi$ i.e., they have no common elements

4. Mutually disjoint

A collection of sets is called a disjoint collection if every pair of sets in the collection of the 2 sets are disjoint. The elements of a disjoint collection are said to be

mutually disjoint.

eg: If $A_1 = \{\{1,2\}, \{3\}\}$ $A_2 = \{\{1\}, \{2,3\}\}$ and $A_3 = \{1,2,3\}$

s.t. A_1, A_2, A_3 are mutually disjoint.

$$A_1 \cap A_2 = \phi, \quad A_1 \cap A_3 = \phi, \quad A_2 \cap A_3 = \phi.$$

$\therefore$ mutually disjoint.

5. Difference (Relative Complement)

The difference of two sets A & B denoted by $A-B$ or A/B is the set of all elements in A which are not in B .

$$A-B = \{x: x \in A \text{ \& } x \notin B\}$$

$$B-A = \{x: x \in B \text{ \& } x \notin A\}$$

Complement (Absolute Complement)

Let U be the universal set. For any set A the relative complement (difference) of A wr.to U i.e., $U-A$ is called the absolute complement of A or the complement of A and is denoted by A' or A^c .

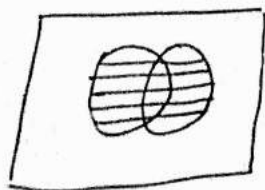
$$U-A = A^c = \{x: x \in U \text{ \& } x \notin A\}$$

Eg: $U = \{1, 2, \dots, 10\}$ & $A = \{2, 4, 6, 8, 10\}$ Then

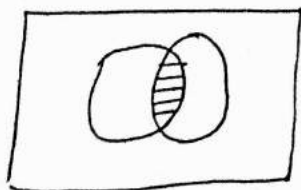
$$A^c = \{1, 3, 5, 7, 9\}$$

Note: $A-B \neq B-A$

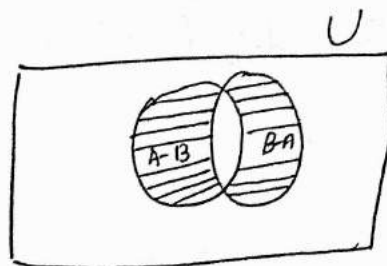
Venn Diagrams of sets operation

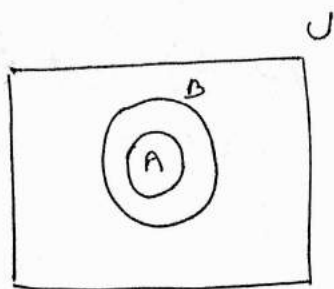


$A \cup B$

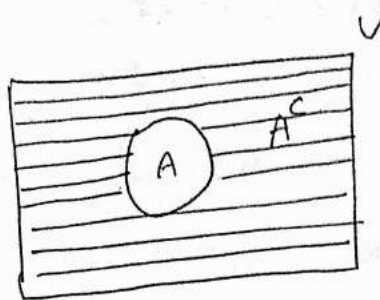


$A \cap B$





ACB



Basic laws of Set Algebra:-

1. Idempotent laws: $A \cup A = A$
For any set A, $A \cap A = A$
2. Commutative laws $A \cup B = B \cup A$
(If A and B are any 2 sets), $A \cap B = B \cap A$
3. Associative laws: $A \cup (B \cap C) = (A \cup B) \cap C$
4. Distributive laws. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
 $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
 $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. Absorption law:
 $A \cup (A \cap B) = A$
 $A \cap (A \cup B) = A$
6. Complement and De Morgan's law.
 $(A \cup B)^c = A^c \cap B^c$ $\phi' = U$ $(A')' = A$
 $(A \cap B)^c = A^c \cup B^c$ $U' = \phi$
- (7) $\downarrow$ $A \cup \phi = A$ $A \cap U = A$ $A \cap \phi = \phi$ $A \cup U' = U$ $A \cap A' = \phi$

7. [Identity laws]

Functions:-

Let X and Y be any two sets. A relation $f: X \rightarrow Y$ is called a function if for every $x \in X$ there is a unique $y \in Y$ such that $f(x) = y$.

The set X is called the domain of the function f and Y is called the co-domain of f .

If x is an element of the set X then the elt $y \in Y$ which is assigned to x is called the image of x so that we have $f(x) = y$. Functions are also called mappings.

Two functions $f: X \rightarrow Y$ & $g: X \rightarrow Y$ are said to be same and we write $f = g$ if $f(x) = g(x) \quad \forall x \in X$.

Eg:- Let $A = \{1, 2, 3\}$ & $B = \{a, b\}$

Let $f = \{(1, a), (2, a), (3, b)\}$ is a fn: $A \rightarrow B$ because domain of $f = A$ and no two ordered pairs have the same first component.

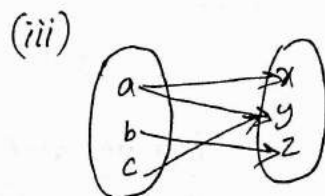
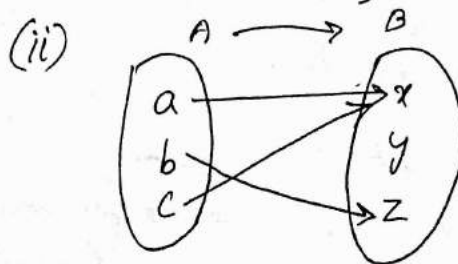
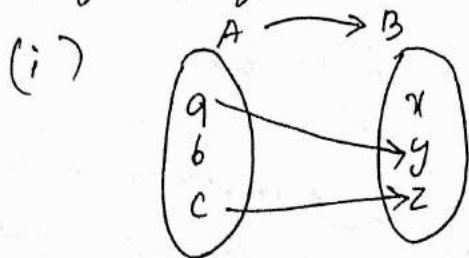
Definition:-

Let $f: X \rightarrow Y$ The range of f consists of those elements in Y which appear as the image of atleast one element in X . we denote range of $f: X \rightarrow Y$ by $f(X)$.

$$R(f) = \{f(x) \in Y : x \in X\}$$

Eg:-

state whether or not each of the diagrams given below defined functions from $A = \{a, b, c\}$ to $B = \{x, y, z\}$.



Solution:-

- (i) not a fn, since nothing is assigned to b
 (iii) not a fn since the elt ' a ' in the domain assigned to 2 elements x and y
 (ii) is a function

Eg: If f is a fn: $\mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 2x + 3$ on the set $\{2, 1, 0, -1, -2\}$ Find the range of f

$$f(x) = 2x + 3.$$

$$f(2) = 7 \quad f(1) = 5 \quad f(0) = 3 \quad f(-1) = 1 \quad f(-2) = -1$$

$$\therefore R(f) = \{5, 3, 1, -1, 7\}$$

Eg: Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$ Let $f = \{(1, a), (1, b), (3, b)\}$

check whether it is a fn or not?

f is not a fn since the element 1 in domain is assigned to 2 elements 'a' and 'b' in the co-domain.

Types of Functions:-

1. One-one function (Injective)

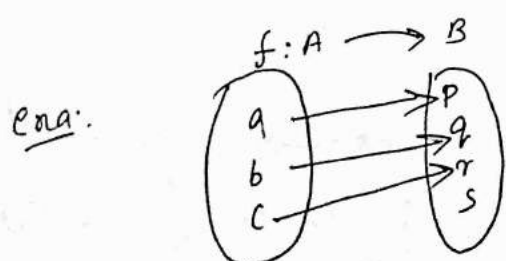
A fn $f: A \rightarrow B$ is said to be one to one function if different elements in A have different images in B .

$$f: A \rightarrow B \text{ is 1-1 iff } f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

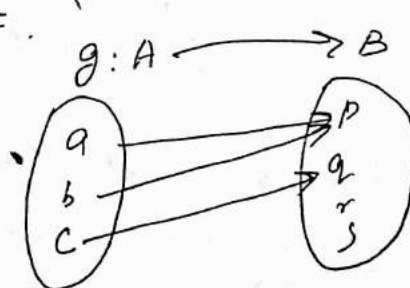
exa: $f: \mathbb{N} \rightarrow \mathbb{N}$ s. $f(x) = 4x + 5$ is 1-1 since

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

$$\text{i.e., } 4x_1 + 5 = 4x_2 + 5 \Rightarrow 4x_1 = 4x_2 \\ x_1 = x_2$$



one-one function



not a 1-1 fn

2. Many one function

A fn $f: A \rightarrow B$ is said to be many one function if two or more elements of A have the same image in B .

3. Onto function (Surjective function)

A fn $f: A \rightarrow B$ is said to be an onto fn or surjective

If each element in B has at least one pre-image in A .
 i.e., if $f: A \rightarrow B$ is an onto fn, then corresponding to each $y \in B$ $\exists$ at least one $x \in A$ such that $y = f(x)$
 $f: A \rightarrow B$ is onto fn if $R(f) = B$.

4. Into function

A fn $f: A \rightarrow B$ is said to be an into function if there exist at least one element in B having no pre-image in A . In other words a fn $f: A \rightarrow B$ is an into function if $R(f) \neq B$.

5. Bijective function

A fn $f: A \rightarrow B$ is said to be a bijective function if it is both one-one and onto.

6. Identity function

Let A be any set. The function $I_A: A \rightarrow A$ defined by $I_A(x) = x \quad \forall x \in A$ is called the identity fn on A .

Let $A = \{a, b, c\}$ then $I_A = \{(a, a), (b, b), (c, c)\}$.

7. Inverse function

Let $f: X \rightarrow Y$ be one to one correspondence from a set X to a set Y . Then inverse fn of f is the fn that assigns to an element $y \in Y$, the unique element $x \in X$ there exist such that $f(x) = y$. The inverse fn is denoted by f^{-1} . Hence $f^{-1}(y) = x$ when $f(x) = y$.

Clearly when $f: X \rightarrow Y$ then $f^{-1}: Y \rightarrow X$. If a function f is not one to one, we cannot define our inverse fn of this function.

Exa: Let $A = \{a, b, c\}$ $B = \{1, 2, 3\}$ and $f = \{(a, 2), (b, 1), (c, 3)\}$ clearly

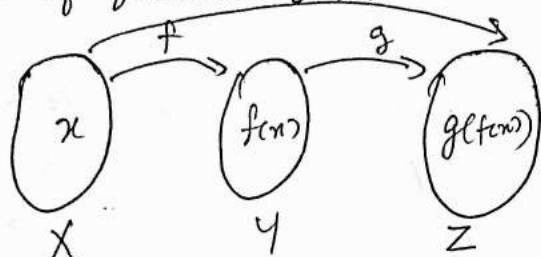
f is a bijection.

$$f^{-1} = \{(2, a), (1, b), (3, c)\}.$$

The fn $f(x) = x^2$ is not invertible since f is not 1-1.

Composition of functions:-

Let $f: X \rightarrow Y$ & $g: Y \rightarrow Z$ be two functions. The composite for $g \circ f: X \rightarrow Z$ defined by $(g \circ f)(x) = g(f(x))$ is called the composition of functions. ~~got~~



Notes:-

1. $R(f)$ is a subset of $D(g)$ which is Y . i.e., $R(f) \subseteq D(g)$ otherwise $g \circ f$ is empty.
2. Given $f: X \rightarrow Y$ & $g: Y \rightarrow Z$ we have the composite for $g \circ f$. However the composite function ~~got~~ $f \circ g$ may or may not exist. For the existence of $f \circ g$ it is necessary that $R_g \subseteq D_f$.
3. $f \circ g \neq g \circ f$ ($f \circ g$ need not be equal to $g \circ f$)

Binary Operations:-

An operation or composition on a non empty set X is a rule that assigns to every ordered n -tuple of elements of X a definite element. When the rule requires only two elements we call this a binary operation.

problems:-

1. State and prove De Morgan's laws:-

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

$$\text{Let } x \in (A \cup B)'$$

$$\Leftrightarrow x \notin A \cup B$$

$$\Leftrightarrow x \notin A \text{ and } x \notin B$$

$$\Leftrightarrow x \in A' \text{ and } x \in B'$$

$$\Leftrightarrow x \in A' \cap B'$$

$$\therefore \underline{(A \cup B)' = A' \cap B'}$$

$$\text{Let } x \in (A \cap B)'$$

$$\Leftrightarrow x \notin A \cap B$$

$$\Leftrightarrow x \notin A \text{ or } x \notin B$$

$$\Leftrightarrow x \in A' \text{ or } x \in B'$$

$$\Leftrightarrow x \in A' \cup B'$$

$$\therefore \underline{(A \cap B)' = A' \cup B'}$$

2. If $A \subseteq B$ then P.T $A \cup B = B$ and $A \cap B = A$ (absorption laws)

$$\text{clearly } B \subseteq A \cup B \text{ --- (1)}$$

$$\text{Let } x \in A \cup B$$

$$\Leftrightarrow x \in A \text{ or } x \in B$$

$$\Leftrightarrow x \in B \text{ or } x \in B \text{ } (\because A \subseteq B)$$

$$\Leftrightarrow x \in B \text{ in both cases}$$

$$\Leftrightarrow A \cup B \subseteq B. \text{ --- (2)}$$

from (1) & (2)

$$\underline{A \cup B = B}$$

$$\text{clearly } A \cap B \subseteq A \text{ --- (1)}$$

$$\text{Let } x \in A \cap B$$

$$\Leftrightarrow x \in B \text{ } [A \subseteq B]$$

$$\Leftrightarrow x \in A, x \in B$$

$$\Leftrightarrow x \in A \cap B$$

$$\therefore A \subseteq A \cap B \text{ --- (2)}$$

from (1) & (2).

$$\underline{A \cap B = A}$$

3. Show that $(A - B) - C = A - (B \cup C)$

$$\text{Note:- } A - B = \{x: x \in A \text{ and } x \notin B\} = \{x: x \in A \cap B'\}$$

$$\therefore A - B = A \cap B'$$

$$\begin{aligned}
 \text{now } (A-B)-C &= (A \cap B') \cap C' \\
 &\Rightarrow A \cap (B' \cap C') \\
 &\Rightarrow A \cap (B \cup C)' \\
 &= A - (B \cup C)
 \end{aligned}$$

$$\text{Hence } \underline{(A-B)-C = A - (B \cup C)}.$$

$$4. \text{ S.T. } A - (B-C) = \underline{(A-B) \cup (A \cap C)}$$

$$\begin{aligned}
 A - (B-C) &= A - (B \cap C') \\
 &= A \cap (B \cap C')' \\
 &= A \cap (B' \cup C) \\
 &= (A \cap B') \cup (A \cap C) \\
 &= \underline{(A-B) \cup A \cap C}
 \end{aligned}$$

$$5 \quad (A \cup B) - C = \underline{(A-C) \cup (B-C)}$$

$$\begin{aligned}
 (A \cup B) - C &= (A \cup B) \cap C' \\
 &= A \cap C' \cup B \cap C' \\
 &= \underline{(A-C) \cup (B-C)}
 \end{aligned}$$

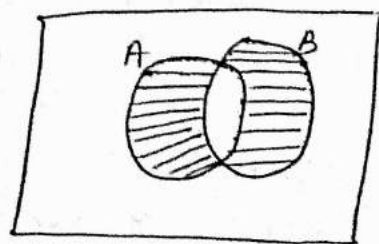
$$6. \quad A - (B \cup C) = (A-B) \cap (A-C)$$

$$\begin{aligned}
 A - (B \cup C) &= A \cap (B \cup C)' \\
 &= A \cap (B' \cap C') \\
 &= (A \cap B') \cap (A \cap C') \\
 &= \underline{(A-B) \cap (A-C)}
 \end{aligned}$$

Defn:- The symmetric difference of sets A and B denoted by $A \Delta B$ or $A \oplus B$ consists of those elements which belongs to A or B but not to both.

$$\text{i.e., } A \Delta B = (A-B) \cup B-A \quad \text{or}$$

$$A \Delta B = (A \cup B) - (A \cap B)$$



$A \Delta B$ is shaded portion

(9) P.T $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$ OR prove that
 $(A \cup B) - (A \cap B) = A \Delta B.$

Consider the result $A - B = A \cap B^c$

$$\begin{aligned} \therefore (A \cup B) - (A \cap B) &= (A \cup B) \cap (A \cap B)^c \\ &= (A \cup B) \cap (A^c \cup B^c) \\ &= (A \cap A^c) \cup (A \cap B^c) \cup (B \cap A^c) \cup (B \cap B^c) \\ &= \phi \cup (A \cap B^c) \cup (B \cap A^c) \cup \phi \\ &= A \cap B^c \cup B \cap A^c \\ &= (A - B) \cup (B - A) \\ &= \underline{\underline{A \Delta B}} \end{aligned}$$

Composition of Functions:-

If $f: A \rightarrow B$ and $g: B \rightarrow C$ then the composition of f and g is a new function from A to C denoted by $g \circ f$ is given by $(g \circ f)(x) = g\{f(x)\}$ for all $x \in A$.

properties:-

1. Composition of functions is associative. i.e., if $f: A \rightarrow B$, $g: B \rightarrow C$ and $h: C \rightarrow D$ are functions then

$$(h \circ g) \circ f = h \circ (g \circ f)$$

Proof:- Since $f: A \rightarrow B$, $g: B \rightarrow C$ then $g \circ f: A \rightarrow C$

Since $g \circ f: A \rightarrow C$ and $h: C \rightarrow D$ then $h \circ (g \circ f): A \rightarrow D$

Similarly since $f: A \rightarrow B$ and $h \circ g: B \rightarrow D$ then

$$(h \circ g) \circ f: A \rightarrow D$$

Thus the domain and co-domain of $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are the same.

Let $x \in A$, $y \in B$ and $z \in C$ then $y = f(x)$ and $z = g(y)$

Then $(g \circ f)(x) = g\{f(x)\} = g(y) = z$

$$h \circ (g \circ f)(x) = \underline{\underline{h(z)}}$$

Also $(h \circ g) \circ f(x) = h \circ g\{f(x)\}$
 $= h\{g(f(x))\}$
 $= h\{g(y)\} = h(z)$

$$\therefore \underline{\underline{(h \circ g) \circ f = h \circ (g \circ f)}}$$

ex:- Let the functions f and g defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$. Find the formula defining the composition function $g \circ f$.

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(2x + 1) \\ &= (2x + 1)^2 - 2 \\ &= 4x^2 + 4x + 1 - 2 \\ &= \underline{\underline{4x^2 + 4x - 1}} \end{aligned}$$

(Q) Consider the function $f: A \rightarrow B$ and $g: B \rightarrow C$ then prove the following:

- (a) If f and g are one to one, then $g \circ f$ is one to one.
 (b) If f and g are onto, then $g \circ f$ is onto.

Proof:- Let f and g are 1-1. So that

$$f(x) = f(y) \Rightarrow x = y$$

$$g(x) = g(y) \Rightarrow x = y.$$

now consider, $(g \circ f)(x) = (g \circ f)(y)$

$$\Rightarrow g\{f(x)\} = g\{f(y)\} \quad \text{since } g \text{ is 1-1}$$

$$\Rightarrow f(x) = f(y) \quad \text{since } f \text{ is 1-1}$$

$$\Rightarrow x = y$$

Hence $g \circ f$ is one-one.

(b) we have $f: A \rightarrow B$ and $g: B \rightarrow C$ so $g \circ f: A \rightarrow C$

To prove $g \circ f$ is onto, we have to prove that
For an element $c \in C$ such that, $(g \circ f)(a) = c \quad \forall c \in C$

Let c be any arbitrary element of C . Since g is onto
there exist a $b \in B$ such that $g(b) = c$. Since
 f is onto there exist an $a \in A$ such that $f(a) = b$.

$$\begin{aligned} \text{But then } (g \circ f)(a) &= g(f(a)) \\ &= g(b) \\ &= c \end{aligned}$$

Hence for each $c \in C$ has a pre image $a \in A$.

So $g \circ f$ is onto.

(18) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 2x - 3$. Find a formula
for f^{-1} .

clearly f is one-one and onto. Hence f^{-1} exists.

Let y be the image of x under f .

$$y = f(x) = 2x - 3. \quad \text{ie, } y = 2x - 3.$$

$$y + 3 = 2x.$$

$$\therefore x = \frac{y+3}{2}$$

consequently x will be the image of y under f^{-1} .

$$\text{so } x = \frac{y+3}{2} \Rightarrow f^{-1}(y) = \frac{y+3}{2}$$

Replace y by x to obtain $f^{-1}(x) = \frac{x+3}{2}$ which
is the formula for f^{-1} .

Identity function:-

Let A be an arbitrary non empty set. The identity on A , denoted by I_A is defined by $I_A(a) = a$.

Theorem:- Let $f: A \rightarrow B$ be any function. Then

$$(a) I_B \circ f = f \quad (b) f \circ I_A = f$$

If f is a one-one correspondence between A and B then

$$(c) f^{-1} \circ f = I_A$$

$$(d) f \circ f^{-1} = I_B.$$

Proof:- $f: A \rightarrow B$ is a function. Then

$$\begin{aligned} (I_B \circ f)(a) &= I_B(f(a)) \\ &= f(a) \text{ for all } a \text{ in } \text{Dom}(f) \end{aligned}$$

∴, $(I_B \circ f)(a) = f(a)$ for all $a \in A$. Hence

$$\underline{I_B \circ f = f} \quad \left[\begin{array}{l} \text{Theorem:-} \\ \text{Let } R \text{ and } S \text{ be relations from } A \text{ to } B \\ \text{if } R(a) = S(a) \text{ for all } a \in A \text{ then } R = S \end{array} \right]$$

$$\begin{aligned} (b) (f \circ I_A)(a) &= f\{I_A(a)\} \\ &= f(a) \quad \text{so } \underline{f \circ I_A = f} \end{aligned}$$

Suppose f is a one-one correspondence between A and B . Then we know that f is invertible. So if $b = f(a)$ then

it is equivalent to $a = f^{-1}(b)$

we have $I_A(a) = a$ for all a in A .

$$\begin{aligned} (c) \text{ so } I_A(a) &= a = f^{-1}\{f(a)\} \\ &= (f^{-1} \circ f)(a) \end{aligned}$$

$$\text{Thus } \underline{I_A = f^{-1} \circ f}$$

$$\begin{aligned} \textcircled{d} \text{ For all } b \in B \quad I_B(b) &= b \\ &= f(f^{-1}(b)) \\ &= (f \circ f^{-1})(b) \end{aligned}$$

$$\text{Thus } I_B(b) = f \circ f^{-1}(b) \quad \forall b \in B$$

$$\text{Hence } \underline{I_B = f \circ f^{-1}}$$

Theorem:- @ Let $f: A \rightarrow B$ and $g: B \rightarrow A$ be functions such that $g \circ f = I_A$ and $f \circ g = I_B$. Then f is one-one correspondence between A and B , g is a one-one correspondence between B and A and each is the inverse of the other.

(b) Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be invertible. Then $g \circ f$ is invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

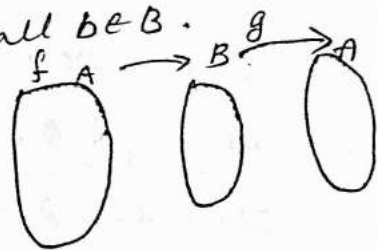
Proof:- Given $f: A \rightarrow B$ and $g: B \rightarrow A$ also

$$g \circ f = I_A \text{ means } (g \circ f)(a) = I_A(a)$$

$$g\{f(a)\} = a \quad \text{for all } a \in A$$

$$f \circ g = I_B \Rightarrow f\{g(b)\} = b. \quad \text{for all } b \in B.$$

This shows that $R(f) = B$ and $R(g) = A$ so each



function f, g is onto.

$$\text{If } f(a_1) = f(a_2) \text{ then } a_1 = g\{f(a_1)\} = g\{f(a_2)\}$$

$$\text{ie, } a_1 = a_2.$$

Thus f is one-to-one. Similarly we can s.t. g is one-one.

So both f and g are invertible.

Now to prove each is the inverse of the other

clearly f^{-1} is defined everywhere since,
 $\text{domain}(f^{-1}) = \text{Range}(f) = B$.

If b is any element in B then,

$$\begin{aligned} f^{-1}(b) &= f^{-1}(f\{g(b)\}) \\ &= (f^{-1} \circ f)g(b) \\ &= I_A(g(b)) \end{aligned}$$

$$\text{since } f^{-1} \circ f = I_A$$

$$\text{since } I_A(a) = a, \forall a \in A$$

$$f^{-1}(b) = g(b)$$

$$\text{Hence } \underline{f^{-1} = g}$$

$$\text{Also } f = (f^{-1})^{-1} = g^{-1} \quad \text{Thus } \underline{g^{-1} = f}$$

Since g and f are onto, f^{-1} and g^{-1} are onto.

Relations:-

A relation R from a set A to a set B is a subset of $A \times B$.
 i.e., if R is a relation from A ~~to~~ ^{to B} then R is a set of ordered pairs where each first element comes from A and each second element comes from B .

$(a, b) \in R$ means a is related to b .

exa: ① Let $A = \{1, 2, 3\}$ $B = \{x, y, z\}$ and let $R = \{(1, y), (1, z), (3, y)\}$

Then R is a relation from A to B , since R is a subset of

$A \times B$. Domain of R is $\{1, 3\}$

Range of R is $\{y, z\}$

exa: ② The set $\{(x, y) \in \mathbb{R}^2 : x < y\}$ is a relation defined on $\mathbb{R}$.

Then $x R y$ iff $x < y$.

Properties :- (Types of Relations)

(1) Let A be a non empty set and R be a binary relation in A . R is said to be Reflexive if aRa for every $a \in A$.

i.e., $(a,a) \in R$ for every $a \in A$.

(2) Symmetric

A relation R on X is said to be symmetric if

$$aRb \Rightarrow bRa \quad \forall a, b \in A \text{ and } B$$

$$\text{i.e., } (a,b) \in R \Rightarrow (b,a) \in R \quad \forall a, b \in A$$

(3) Transitive

A relation R on X is called transitive if aRb and bRc

then $aRc \quad \forall a, b, c \in A$

$$\text{i.e., } (a,b) \text{ and } (b,c) \in R \Rightarrow (a,c) \in R.$$

(4) Anti symmetric

A relation R is called anti symmetric if aRb and bRa implies $a=b$.

Equivalence Relation

A relation R is called an equivalence relation if it is reflexive, symmetric and transitive.

Exa: Let $A = \{1, 2, 3, 4\}$ $R = \{(1,1)(1,2)(2,1)(1,3)(3,1)(2,2)(3,3)\}$

$$S = \{(1,1)(1,2)(1,3)(2,1)(2,2)(2,3)(3,1)(3,2)(3,3)\}$$

Then R is symmetric but not reflexive as $(4,4) \notin R$.

Also not transitive.

S is ^{not} reflexive but symmetric & transitive.

exa: The relation $\leq$ is reflexive in the set of all real numbers.

but the relation $<$ is not reflexive in the set of real nos.

exa: The relation 'divides' is asymmetric.

Ques: Let m be a +ve integer greater than 1. Show that the relation $R = \{(a, b) / a \equiv b \pmod{m}\}$ is an equivalence relation on the set of integers.

Proof:- $a \equiv b \pmod{m}$ means $a - b$ is divisible by m .

So if $(a, b) \in R$ then $a \equiv b \pmod{m}$.

clearly R is reflexive

since $a R a$ i.e. $a \equiv a \pmod{m}$

$a - a = 0$ is divisible by m .

Now let $(a, b) \in R$ i.e. $a \equiv b \pmod{m}$ Then $(a - b)$ is divisible by m . So $(a - b) = km$, where k is an integer.

$$(b - a) = -km$$

So $b \equiv a \pmod{m} \Rightarrow (b, a) \in R$.

So R is symmetric

Let $(a, b) \in R$ $(b, c) \in R$ Then $a \equiv b \pmod{m}$
 $b \equiv c \pmod{m}$.

$$\text{i.e., } (a - b) = km$$

$$(b - c) = lm$$

adding two, $a - c = (k + l)m$

i.e., $a \equiv c \pmod{m}$. Hence $(a, c) \in R$

$\therefore R$ is transitive also. So congruence modulo m

is an equivalence relation.

(Q) Give an example of a relation which is

- (a) reflexive but not symmetric or transitive
- (b) symmetric but not reflexive or transitive (The ones of +)
- (c) transitive but not reflexive or symmetric
- (d) reflexive and symmetric but not transitive
- (e) reflexive and transitive but not symmetric (The ones of +)
- (f) symmetric and transitive but not reflexive.

Ans: If $A = \{1, 2, 3, 4\}$ Then

$$R_1 = \{ (1,1) (2,2) (3,3) (4,4) (1,3) (3,2) \}$$
 is reflexive but not symmetric

or transitive.

$$R_2 = \{ (1,1) (1,3) (3,1) (3,4) (4,3) \}$$
 is symmetric but not reflexive or transitive.

$$R_3 = \{ (1,1) (1,3) \}$$
 is transitive.

$$R_4 = \{ (1,1) (2,2) (3,3) (4,4) (1,3) (3,1) (3,4) (4,3) \}$$
 is not transitive

$R_5 =$ The relation of ' $\subset$ ' is an example ^{on} which R is reflexive and transitive but not symmetric.

$$R_5 = \{ (1,1) (2,2) (2,3) (3,2) (3,3) \}$$
 is not reflexive.

Partition of a set:-

A partition of X is a disjoint class $\{x_i\}$ of non empty subsets of X whose union is the full set X itself. The x_i 's are called the partition sets.

Exa: If $X = \{1, 2, 3, 4, 5\}$ then $\{1, 3, 5\}$ $\{2, 4\}$ and $\{1, 2, 3\}$ $\{4, 5\}$ are two different partitions

of X .

If X is the set of all real numbers then we can partition X into the set of all rationals and the set of all irrationals.

Equivalence class:-

Suppose R is an equivalence relation on a set S . For each $a \in S$, let $[a]$ denote the set of elements of S to which a is related under R .

i.e., $[a] = \{x : (a, x) \in R\}$ we call $[a]$ the equivalence class of a in S .

ex: let $S = \{1, 2, 3\}$ and R be a relation on S

$$R = \{(1, 1), (1, 2), (2, 1), (3, 3), (2, 2)\}.$$

Clearly R is an equivalence relation.

$$[1] = \{1, 2\} \quad [2] = \{1, 2\} \quad [3] = \{3\}. \quad \text{Therefore}$$

the set $\{[1], [3]\}$ is a partition of S .

ex:- let R be a relation on the set Z of integers defined by $x \equiv y \pmod{5}$. Here the equivalence classes are

$$[0] = A_0 = \{\dots, -10, -5, 0, 5, 10, \dots\}$$

$$[1] = A_1 = \{\dots, -9, -4, 1, 6, 11, \dots\}$$

$$[2] = A_2 = \{\dots, -8, -3, 2, 7, 12, \dots\}$$

$$[3] = A_3 = \{\dots, -7, -2, 3, 8, 13, \dots\}$$

$$[4] = \{\dots, -6, -1, 4, 9, 14, \dots\}$$

$$[5] = \{\dots, -5, 0, 5, 10, 15, \dots\} = \underline{\underline{[0]}}.$$

$$\therefore Z = [0] \cup [1] \cup [2] \cup [3] \cup [4] \quad \text{which is a partition}$$

of Z .

Partial Ordering Relations

A relation R on a set is called a partial ordering or partial order if R is reflexive, antisymmetric and transitive.

A set S together with a partial ordering is called a partially ordered set or poset.

exa:-1 The relation $\subseteq$ of set inclusion is a partial ordering on any collection of sets. since ' $\subseteq$ ' set inclusion has 3 desired properties.

- (1) $A \subseteq A$ for any set A
- (2) If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- (3) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

exa:-2 The relation $\leq$ on the set R of real numbers is a partial ordering

exa:-3 The relation 'a divides b' is a partial ordering on the set N of positive integers.

But the relation 'a divides b' is not a partial ordering on the set of Z integers.

for exa: $3/-3$ and $-3/3$ but $3 \neq -3$. (not antisymmetric)

Module III (Set Theory)

Equivalence relation.

(V.8) Let R denote a relation on the set of ordered pairs of positive integers such that $(x, y) R (u, v)$ iff $xv = yu$. S.T. R is an equivalence relation.

R is an equivalence reln. iff it satisfies reflexive, symmetric & transitive properties.

clearly R is reflexive,

since $(x, y) R (x, y)$ for every ordered pair (x, y)

ie, $x y = y x$ always.

So R is reflexive.

Let $(x_1, y_1) R (x_2, y_2)$ then

$$\text{we have } x_1 y_2 = y_1 x_2$$

$$y_2 x_1 = x_2 y_1$$

$$\text{ie, } x_2 y_1 = y_2 x_1 \quad \therefore (x_2, y_2) R (x_1, y_1)$$

R is symmetric

Let $(x_1, y_1) R (x_2, y_2)$ and $(x_2, y_2) R (x_3, y_3)$

then we have,

$$x_1 y_2 = y_1 x_2 \quad \text{and} \quad x_2 y_3 = y_2 x_3 \quad \text{--- (2)}$$

$$\frac{y_2}{x_2} = \frac{y_1 x_2}{x_1} \quad \therefore \quad x_2 y_3 = \frac{y_1 x_2}{x_1} \cdot x_3$$

Substituting in (2) $\Rightarrow$

$$\text{i.e., } y_3 = \frac{y_1 x_3}{x_1}$$

$$x_1 y_3 = y_1 x_3$$

$$\text{i.e., } (x_1, y_1) R (x_3, y_3)$$

$\therefore R$ is transitive.

Hence R is an equivalence relation.

(8) Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and a relation

R defined as $R = \{(x, y) : |x - y| = 2\}$

is R an equivalence relation.

$$R = \{(1, 3), (2, 4), (3, 5), (4, 6), (5, 7), (7, 5), (6, 4), (5, 3), (4, 2), (3, 1)\}$$

R is not reflexive since $(x, x) \notin R$.

R is symmetric since $(x, y) \in R \Rightarrow (y, x) \in R$
for every (x, y) .

R is not transitive $(x, y), (y, z) \in R$
but $(x, z) \notin R$.

Hence R is not an equivalence relation.

8. If R is a relation on the set of positive integers such that $(a, b) \in R$ iff $a^2 + b$ is even. prove that R is an equivalence relation.

To prove R is reflexive.

[If $(a, a) \in R$ then we should have
 $a^2 + a$ is even]

$a^2 + a = a(a+1)$ since a and $(a+1)$ are consecutive positive integers.
= even number

Let $(a, b) \in R$ then $a^2 + b$ is even.
 $\Rightarrow a$ and b ~~are~~ must be both even
or both odd.

in either case $b^2 + a$ will be even.

ie, $b^2 + a$ is even $\Rightarrow b R a$

$\therefore (b, a) \in R$ Hence

R is symmetric

Let $a R b$ and $b R c$ then

$a^2 + b$ is even & $b^2 + c$ even

a, b are ^{either} both even
or both odd

b, c are ^{either} both even

b, c are both odd.

When a, c are both even then
clearly $a^2 + c$ will be even.

When a, c are both odd then also
 $a^2 + c$ will be even.

Hence $a R c$. $\therefore$ transitive.

$\therefore R$ is an equivalence relation.

8) Let R be defined on the set of real numbers
by if x, y are real numbers,
 $x R y \Leftrightarrow x - y$ is a rational number

8.7 the relation R is an equivalence relation

given. $x R y$ iff $x - y$ is a rational number.

$x R x$ since $x - x = 0$ is a rational number.

$\therefore$ reflexive

Let $x R y$ then $x - y = z$ for some rational number.

$y - x = -z$ is also a rational no.

$\therefore y R x$. $\therefore$ Symmetric

Let $x R y$ and $y R z$ then

$x - y = r_1$ and $y - z = r_2$ are rational numbers.

Now, $x - y + y - z = r_1 + r_2$

$\Rightarrow x - z = r_1 + r_2$ is also a rational number.

So $x R z$

$\therefore R$ is transitive. Hence it is an equivalence relation.

Antisymmetric Relation

Let R be a relation on a set A . Then R is said to be antisymmetric relation if aRb and $bRa \Rightarrow a=b \quad \forall a, b \in A$
 i.e., If $(a, b) \in R$ and $(b, a) \in R$
 then $a=b \quad \forall a, b \in R$.

Exa: Let $A = \{4, 5, 6\}$ and

$R = \{(4, 4), (4, 5), (5, 4), (5, 6), (4, 6)\}$ is the relation R antisymmetric?

The reln R is not antisymmetric as $4 \neq 5$ ~~since~~ ^{but} $(4, 5)$ and $(5, 4)$ belongs to R .

The relation R is antisymmetric as $a=b$ when (a, b) and (b, a) both belongs to R .

Exa: The relation of divisibility on N is antisymmetric since whenever m is divisible by n and n is divisible by m

then $m=n$.

Congruence

Let n be a +ve integer and a, b are integers. we define $a \equiv b \pmod{n}$ if $a-b$ is a multiple of n .

Equivalence class

If R is an equivalence relation on A , ^{as a} ~~the~~ the set of all elements of A that are related to an element a of A is called the equivalence class of a and denoted by $[a]$.

In other words,

$$[a] = \{x \mid x R a\}$$

Exa: Let $A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$ is an equivalence relation on A . Then

$$[1] = \{1, 2\} \quad [3] = \{3\}$$

$$[2] = \{1, 2\}$$

- Q3) On the set of the integers congruent modulus is an equivalent relation prove it.
Find the equivalent class.

We have given, $a \equiv b \pmod{5}$

ie, $(a-b)$ is multiple of 5

$$a-b = m(5)$$

1. $a \equiv a \pmod{5}$ since $a-0=0$

We can write $0 = m(5)$. So, aRa

$\therefore R$ is reflexive.

2. Let aRb then $a \equiv b \pmod{5}$

$$\text{Or } a-b = m(5)$$

$$\Rightarrow -(a-b) = m(5)$$

$$\Rightarrow b-a = m(5)$$

$$\Rightarrow b \equiv a \pmod{5}$$

$$\Rightarrow bRa$$

$\therefore R$ is symmetric

3. Let aRb & bRc

$$\Rightarrow a-b = m(5) \quad \& \quad b-c = n(5)$$

$$= 5m \qquad \qquad \qquad = 5n$$

then $a - c = 5m + 5n$
 $= 5(m+n)$

ie, $a - c$ is a multiple of 5

or $a \equiv c \pmod{5}$

$\therefore$ Transitive. Hence R is an equivalence relation

equivalent classes

$[0] = \{x : x R 0\}$ ie, all x should be multiples of 5

$[0] = \{\dots, -10, -5, 0, 5, 10, 15, \dots\}$

$[1] = \{\dots, -9, -4, 1, 6, 11, 16, \dots\}$

$[2] = \{\dots, -8, -3, 2, 7, 12, 17, \dots\}$

$[3] = \{\dots, -7, -2, 3, 8, 13, 18, \dots\}$

$[4] = \{\dots, -6, -1, 4, 9, 14, 19, \dots\}$

Q3 S.T the set of integers congruent modulo 6 is an equivalent relation & find the equivalent class.

Q. If R is a relation on the set of integers such that $\langle a, b \rangle \in R$ if $3a + 4b = 7n$ for some n . Prove that R is an equivalence relation.

1. Clearly aRa since,

$$3a + 4a = 7a \quad \therefore \langle a, a \rangle \in R$$

R is reflexive.

2. Let aRb then $3a + 4b = 7n$ (multiple of 7)

$$\text{Now } 7(a+b) - (3a + 4b) = 4a + 3b = m(7)$$

$$\Rightarrow 3b + 4a = m(7) \text{ a multiple of } 7$$

$$\therefore \langle b, a \rangle \in R \quad bRa$$

$\therefore R$ is symmetric

3. If aRb & bRc then

$$3a + 4b = 7n \quad \& \quad 3b + 4c = 7m \quad \text{for some } n, m \in \mathbb{Z}$$

adding

$$3a + 4b + 3b + 4c = 7(n+m)$$

$$3a + 4c = 7(n+m-b)$$

$$3a + 4c = m'(7) \text{ is a multiple of } 7$$

$\therefore$ Transitive and it is an equivalence relation.

Domain & Range

Let R be a relation from set A to set B then the domain of R is denoted and defined by $D(R) = \{a : \langle a, b \rangle \in R\}$

Range of R is defined as

$$\text{Range}(R) = \{b : \langle a, b \rangle \in R\}$$

Exa: $A = \{1, 2, 3\}$ $B = \{a, b, c\}$

$$R = \{(1, a), (2, b), (3, a)\}$$

$$D(R) = \{1, 2, 3\}$$

$$\text{Range}(R) = \{a, b\}$$

Inverse

from $A \rightarrow B$
 $R: A \rightarrow B$

Let R a relation

$$R = \{(a, b) : a \in A, b \in B\}$$

then the inverse of R is denoted and defined by

$$R^{-1} = \{(b, a) : a \in A, b \in B\}$$

Exa 2 $A = \{1, 2, 3\}$ $B = \{a, b, c\}$

$$R = \{(1, a), (2, b), (3, a)\}$$

$$R^{-1} = \{(a, 1), (b, 2), (a, 3)\}$$

Partition

Let S be a given set when $A = \{A_1, A_2, \dots, A_n\}$

Where each A_i $i = 1, 2, 3, \dots, n$ is a subset of S

Then A is called a partition of S if each A_i is mutually disjoint and $\bigcup_{i=1}^n A_i = S$.

Exa: Let $S = \{1, 2, 3, 4, 5, 6\}$ and $A = \{A_1, A_2, A_3\}$

where $A_1 = \{1, 2\}$ $A_2 = \{3, 4, 6\}$ $A_3 = \{5\}$

then A forms a partition of S . Since

A_1, A_2, A_3 are mutually disjoint set

and $A_1 \cup A_2 \cup A_3 = S$.

Exa: 3 $B_1 = \{1, 2, 3\}$ $B_2 = \{3, 4, 5\}$, $B_3 = \{6\}$ Then

B_1, B_2, B_3 are not a partition of S .

Since $B_1 \cap B_2 = \{3\} \neq \phi$.

$\{\{1, 2, 3\}, \{4, 5\}\}$ is not a partition of S , since Union is not S .

* If R is an equivalence relation on a set A , then the equivalence classes form a partition of A . We can define an equivalence relation on A for which these equivalence classes are partition of A .

Exa:- $a \equiv b \pmod{5}$ if $(a, b) \in R$.

where a & b are integers. then the equivalence classes are

$$[0] = \{ \dots, -10, -5, 0, 5, 10, 15, \dots \}$$

$$[1] = \{ \dots, -9, -4, 1, 6, 11, 16, \dots \}$$

$$[2] = \{ \dots, -8, -3, 2, 7, 12, 17, \dots \}$$

$$[3] = \{ \dots, -7, -2, 3, 8, 13, 18, \dots \}$$

$$[4] = \{ \dots, -6, -1, 4, 9, 14, 19, \dots \}$$

Clearly $\{[0], [1], [2], [3], [4]\}$
be the partitions of the set.

Function

A fn or mapping defined as $f: X \rightarrow Y$ is a relationship from elements of one set X to elements of set Y .

A relation f from a set X to another set Y is called a function if for every $x \in X$ there is a unique $y \in Y$ such that $y = f(x)$.

If $y = f(x)$, x is called an argument or pre image and y is called the image of x under f .

Here X is the domain of f , denoted by D_f and Y is called the co-domain of f .

The set of image of all elements of X is called the range of f denoted by

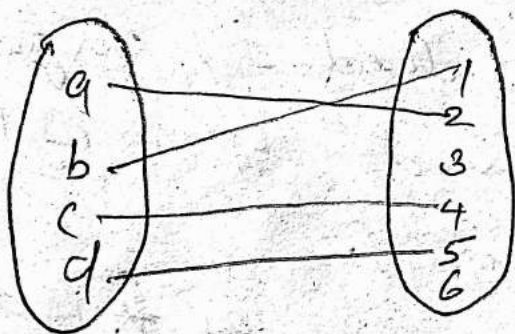
R_f .

$$R_f = \{y \in Y \mid y = f(x) \text{ for some } x \in X\}$$

Types of Functions:-

One-one function (Injective)

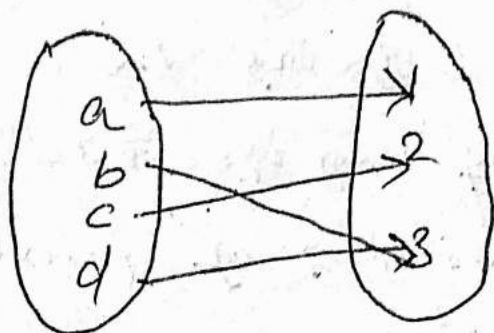
A function $f: X \rightarrow Y$ is called one to one or injective if distinct elements of X are mapped onto distinct element of Y . In other words, f is 1-1 iff, $f(x_1) = f(x_2)$ whenever $x_1 = x_2$.



Onto function (Surjective)

A fn $f: X \rightarrow Y$ is called onto iff for every element $y \in Y$ there is an element $x \in X$ such that $f(x) = y$.

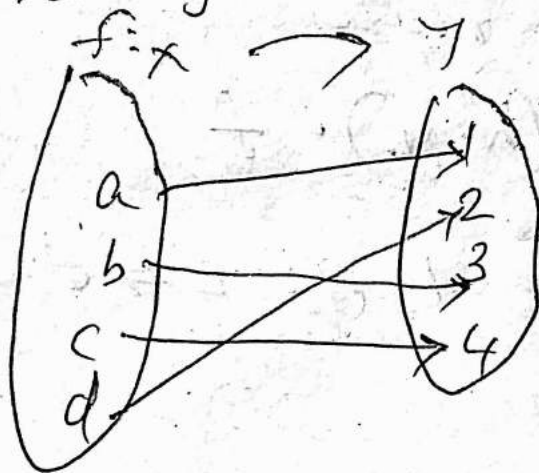
Or f is onto if $R_f = Y$



Bijjective fn:-

A fn $f: X \rightarrow Y$ is called bijective if it is both 1-1 and onto.

If X, Y are finite sets that $f: X \rightarrow Y$ is bijective then X and Y have the same no. of elements.



Composition of functions

If $f: A \rightarrow B$ and $g: B \rightarrow C$ then the composition of f and g is a new fn from $A \rightarrow C$ denoted by $g \circ f$ is given by $(g \circ f)(x) = g(f(x))$ for all $x \in A$

① If f, g are relations then the composition of f and g is denoted by $f \circ g$ whereas the composition of functions f and g is denoted by $g \circ f$.

② When $g \circ f$ is defined, $f \circ g$ need not be defined. To define $g \circ f$, we must have, $\text{range}(f) \subseteq \text{dom}(g)$. Composition of functions is not commutative.

③ If $f: A \rightarrow B$, $g: B \rightarrow C$ and $h: C \rightarrow D$ are functions then

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Proof:

Since $f: A \rightarrow B$ and $g: B \rightarrow C$ then

$$g \circ f: A \rightarrow C$$

Since $g \circ f: A \rightarrow C$ and $h: C \rightarrow D$ then

$$\boxed{h \circ (g \circ f): A \rightarrow D}$$

Now: $f: A \rightarrow B$ and $h \circ g: B \rightarrow D$

$$\boxed{(h \circ g) \circ f: A \rightarrow D}$$

Thus the domain and co-domain of $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are same.

Let $x \in A$, $y \in B$ and $z \in D$ so that
 $y = f(x)$ and $z = g(y)$

$$\begin{aligned} \text{Then } (g \circ f)(x) &= g(f(x)) \\ &= g(y) \\ &= z \end{aligned}$$

$$\therefore \{h \circ (g \circ f)\}(x) = h(z)$$

$$h \circ (g \circ f)(x) = h(z) \quad \text{--- (1)}$$

$$\begin{aligned} \text{Also } ((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) \\ &= h\{g(f(x))\} \\ &= h[g(y)] \\ &= h(z) \quad \text{--- (2)} \end{aligned}$$

from (1) and (2) we get

$h \circ (g \circ f) = (h \circ g) \circ f$, since they are true for all $x \in A$.

① Let $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x + 2$
s.t. it is 1-1.

f is 1-1 iff $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$$\text{i.e., } x_1 + 2 = x_2 + 2 \Rightarrow x_1 = x_2$$

$\therefore f$ is 1-1

② $g: \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(x) = x^2$

$$g(x_1) = g(x_2) \Rightarrow x_1^2 = x_2^2$$

$$\Rightarrow \pm x_1 = \pm x_2$$

$$\text{i.e., } x_1^2 = x_2^2 \not\Rightarrow x_1 = x_2$$

hence it is not 1-1

③ Let $f: \mathbb{N} \rightarrow \mathbb{N}$ $f(x) = x^2 + 2$ is it a bijective function.

To check f is 1-1

$$\text{let } f(x_1) = f(x_2) \Rightarrow x_1^2 + 2 = x_2^2 + 2$$

$$\Rightarrow x_1^2 = x_2^2$$

$$\Rightarrow x_1 = x_2$$

So f is ~~not~~ one-one.

To check if f is onto

let $y \in \mathbb{N}$ then $y = x^2 + 2$ for $x \in \mathbb{N}$

$$x^2 = y - 2$$

$$x = \sqrt{y - 2}$$

here $y - 2 \geq 0$ i.e. $y \geq 2$.

When $y = 1$, $x = \sqrt{-1} = \pm i \notin \mathbb{N}$.

So f is not onto.

it is not bijective

⑧ Determine whether or not each of the following defines a 1-1 or onto function with justification.

① $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ defined by $f(m, n) = 2m + 3n$

$$f(m, n) = 2m + 3n \text{ where } (m, n) \in \mathbb{N} \times \mathbb{N}$$

Let us take $m = 1$ and $n = 3$.

$$\text{Then } f(1, 3) = 11$$

When $m = 4$ and $n = 1$ also,

$$f(4, 1) = 11$$

Two ^{different} elements of $\mathbb{N} \times \mathbb{N}$ have the same image. Which is not possible. So it is not a 1-1 fn.

Let us try to find the pre image of $1 \in \mathbb{N}$.

i.e., to find $(m, n) \in \mathbb{N} \times \mathbb{N}$ such that

$$f(m, n) = 2m + 3n = 1$$

Values of m & n do not exist since

$$2m+3n \geq 5, \text{ as } m, n \text{ each } \geq 1$$

Hence $f(m, n)$ is not onto.

(2) $f: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $f(m, n) = 2m + 3n$
 $f(m, n) = 2m + 3n$ where $(m, n) \in \mathbb{Z} \times \mathbb{Z}$

As the above problem,

$$f(1, 3) = 11$$

$$f(4, 1) = 11$$

but $(1, 3) \neq (4, 1) \therefore f$ is not 1-1.

let us try to find the preimage of $k \in \mathbb{Z}$
 i.e., to find $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ such that

$$f(m, n) = 2m + 3n = k$$

if we take
 clearly, $m = -k$ and $n = k$

Thus for all $k \in \mathbb{Z}$, we are able to
 find $(-k, k) \in \mathbb{Z} \times \mathbb{Z}$ which is the
 pre image of k .

Hence $f(m, n)$ is onto.

(Q) Let $X = Y = Z = \mathbb{R}$ be the set of reals. Consider the functions f and g such that
 $f(x) = x + 2$, $x \in \mathbb{R}$ $g(y) = y^2$ $y \in \mathbb{R}$
 find $g \circ f$.

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(x+2) \\ &= \underline{\underline{(x+2)^2}} \quad \text{where } x \in \mathbb{R}. \end{aligned}$$

(Q) Let $f, g: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = x^2 + 3x + 1, \quad g(x) = 2x - 3$$

find (i) $f \circ f$ (ii) $f \circ g$ (iii) $g \circ f$

$$(f \circ f)(x) = f(f(x))$$

$$= f(x^2 + 3x + 1)$$

$$= (x^2 + 3x + 1)^2 + 3(x^2 + 3x + 1) + 1$$

$$= x^4 + 9x^2 + 1 + 6x^3 + 2x^2 + 6x + 3x^2 + 9x + 3 + 1$$

$$= \underline{\underline{x^4 + 6x^3 + 14x^2 + 15x + 5}}$$

$$\begin{aligned}
 (f \circ g)(x) &= f(g(x)) \\
 &= f(2x-3) \\
 &= (2x-3)^2 + 3(2x-3) + 1 \\
 &= \underline{\underline{4x^2 - 6x + 1}}
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad (g \circ f)(x) &= g(f(x)) \\
 &= g(x^2 + 3x + 1) \\
 &= 2(x^2 + 3x + 1) - 3 \\
 &= 2x^2 + 6x + 2 - 3 \\
 &= \underline{\underline{2x^2 + 6x - 1}}
 \end{aligned}$$

5.9) Let f, g, h are functions on integers
 $f(n) = n^2$, $g(n) = n+1$, $h(n) = n-1$
 determine (i) $f \circ g \circ h$ (ii) $g \circ f \circ h$
 (iii) $h \circ f \circ g$

$$\begin{aligned}
 (f \circ g \circ h)(n) &= f(g(h(n))) \\
 &= f(g(n-1)) \\
 &= f(n-1+1) = f(n) = \underline{\underline{n^2}}
 \end{aligned}$$

$$\begin{aligned}
 (g \circ f \circ h)(n) &= g[f(h(n))] \\
 &= g[f(n-1)] \\
 &= g[(n-1)^2] \\
 &= (n-1)^2 + 1 = n^2 - 2n + 1 + 1 \\
 &= \underline{\underline{n^2 - 2n + 2}}
 \end{aligned}$$

~~$$(g \circ f \circ h)(n) = g \circ f(h(n))$$~~

$$\begin{aligned}
 (h \circ f \circ g)(n) &= h(f(g(n))) \\
 &= h[f(n+1)] \\
 &= h[(n+1)^2] \\
 &= (n+1)^2 - 1 \\
 &= n^2 + 2n + 1 - 1 \\
 &= \underline{\underline{n^2 + 2n}}
 \end{aligned}$$

U-8 ① Suppose that $f(x) = x + 2$, $g(x) = x - 2$ and $h(x) = 3x$ for $x \in \mathbb{R}$ where $\mathbb{R}$ is the set of real nos find
 (i) $g \circ f$ (iii) $f \circ f$ (v) $f \circ h$ (vii) $h \circ h$
 (ii) $f \circ g$ (iv) $g \circ g$ (vi) $h \circ g$ (viii) $(f \circ h) \circ g$

(Q) If $A = \{1, 2, 3, 4, 5\}$ $B = \{1, 2, 3, 8, 9\}$

and the functions $f: A \rightarrow B$ and

$g: A \rightarrow A$ are defined by

$$f = \{(1, 8), (3, 9), (4, 3), (2, 1), (5, 2)\}$$

$$g = \{(1, 2), (3, 1), (2, 2), (4, 3), (5, 2)\}$$

find $f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$

if they exist.

★ To define $g \circ f$, $\text{range}(f) \subseteq \text{Dom}(g)$

To define $f \circ g$, $\text{range}(g) \subseteq \text{dom}(f)$

$f \circ f$, $\text{range}(f) \subseteq \text{dom}(f)$

$g \circ g$, $\text{range}(g) \subseteq \text{dom}(g)$

Since $g: A \rightarrow A$ & $f: A \rightarrow B$

$$\boxed{f \circ g: A \rightarrow B}$$

$$(f \circ g)(1) = f(g(1)) = f(2) = 1$$

$$(f \circ g)(2) = f(g(2)) = f(2) = 1$$

$$A = \{1, 2, 3, 4, 5\}$$

$$B = \{1, 2, 3, 8, 9\}$$

$$(f \circ g)(3) = f(g(3)) = f(1) = 8$$

$$(f \circ g)(4) = f(g(4)) = f(3) = 9$$

$$(f \circ g)(5) = f(g(5)) = f(2) = 1$$

To define $g \circ f$; $\text{range}(f) \subseteq \text{domain}(g)$

$$\text{Here } \text{range}(f) = \{1, 2, 3, 8, 9\}$$

$$\text{dom}(g) = \{1, 2, 3, 4, 5\}$$

$$\text{range}(f) \not\subseteq \text{dom}(g)$$

Hence $g \circ f$ is not defined.

$$\text{Again } \text{range}(f) = \{1, 2, 3, 8, 9\}$$

$$\text{dom}(f) = \{1, 2, 3, 4, 5\}$$

Hence $f \circ f$ is not defined.

$$\text{range}(g) = \{1, 2, 3\} \subseteq \text{dom}(g) = \{1, 2, 3, 4, 5\}$$

Hence $g \circ g$ is defined.

Posets, Lattices and Boolean Algebra

Posets:- Consider a relation R on a set S satisfying the following properties

1. R is reflexive $x, x R x \quad \forall x \in S$
2. R is antisymmetric if $x R y, y R x \Rightarrow x = y$
3. R is transitive if $x R y, y R z \Rightarrow x R z$

Then R is called a partially order relation and the set S together with partial order is called a poset or partially order set denoted by $(S, \leq)$

Exa: The set N of natural nos form a poset under the relation ' $\leq$ '

- ② The set N of natural nos under divisibility i.e., ' x divides y ' forms a poset
- ③ Consider a set $S = \{1, 2\}$ and power set of S is $P(S)$. The relation of set inclusion ' $\subseteq$ ' is a partial order

1. The usual order $\leq$ on the set of real no.s is a partial order.

Since

$$a \leq a \quad \forall a \in \mathbb{R}$$

$$a \leq b, b \leq a \Rightarrow a = b$$

$$a \leq b, b \leq c \Rightarrow a \leq c$$

$\therefore \leq$ is reflexive, antisymmetric & transitive.

2. The relation $<$ (less than) on the set of real no.s is not partial order.

$\therefore$ it is not reflexive ($a \nless a$)

3. Set inclusion

$\subset$ (subset) on $\mathcal{P}(S)$ is a partial order (where S is any set)

$$A \subset A \quad \forall A \in \mathcal{P}(S) \quad \text{reflexive}$$

$$A \subset B, B \subset A \Rightarrow A = B \quad \text{antisymmetric}$$

$$A \subset B, B \subset C \Rightarrow A \subset C \quad \text{transitive}$$

$\therefore \subset$ is a partial order.

#. ' $|$ ' (divides) on set of natural no.s is a partial order

$$3 | 6 \quad 3 \nmid 5$$

$$a | a \quad \forall a \in \mathbb{N}$$

$$a | b, b | a \Rightarrow a = b$$

$$a | b, b | c \Rightarrow a | c$$

$\therefore$ is a partial order.

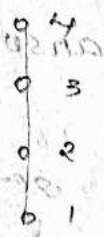
is not partial order in set of integers is not reflexive

" " " " set $\mathbb{Z} \setminus \{0\}$
relation is not antisymmetric. $a | b, b | a \Rightarrow a = \pm b$

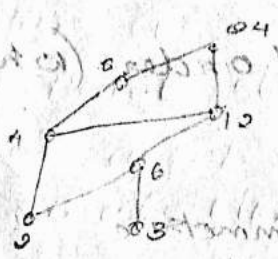
A set with a partial order is called a partially ordered set or poset.

HASSE DIAGRAM for poset:

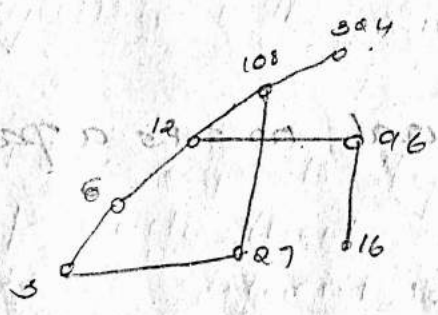
Draw the HASSE Diagram for the set $\{1, 2, 3, 4\}$ with the usual order as the partial order.



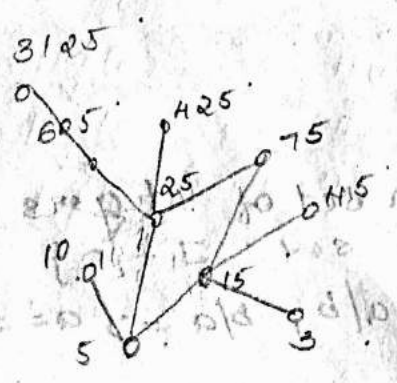
$\{2, 4, 6, 8, 12, 24, 3, \}$ with ' $\mid$ ' as the order.



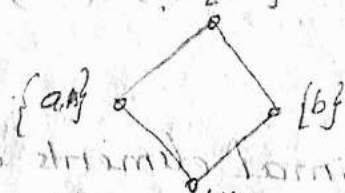
$\{3, 6, 27, 108, 12, 324, 16, 96\}$ with ' $\mid$ ' as order.



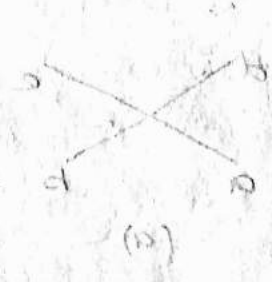
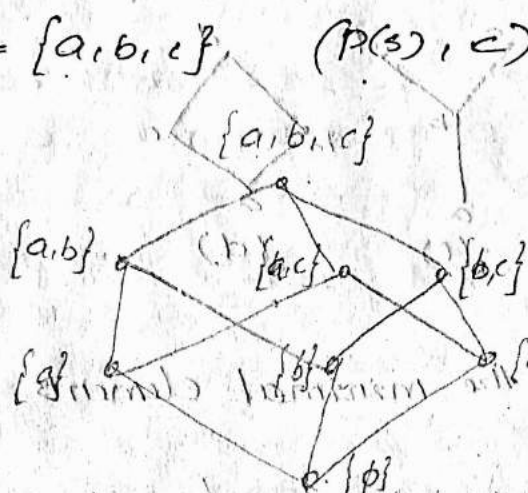
$\{5, 10, 25, 75, 425, 625, 3125, 3, 15, 45\}$



9. Draw Hasse diagram for $(P(S), \subseteq)$ $S = \{a, b\}$.



eg 3 $S = \{a, b, c\}$, $(P(S), \subseteq)$



When $P, \leq$ is a poset. An element $a \in P$ is called a maximal element if there is no element $b \in P$ such that $a < b$, i.e., a strictly precedes b .

There is no element b which strictly succeeds a .

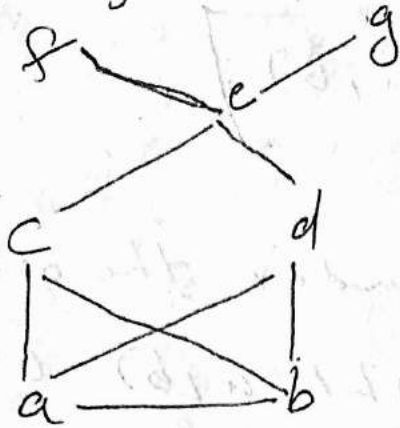
An element $a \in P$ is called a minimal element if there is no element which strictly precedes a .

$b < a$ i.e., There is no element which strictly precedes a .

In eg(2) ϕ is the minimal element.

$\{a, b, c\}$ is the maximal element.

Exa: Consider the poset $A = \{a, b, c, d, e, f, g\}$ Also 207
 let $B = \{c, d, e\}$. determine upper and lower
 bounds of B .



The upper bounds of B are e, f, g since every element of B is $\leq e, f, g$

The lower bounds of B are a, b since a & b are $\leq$ every element of B .

★ Least upperbound = Supremum ($\cup$ LUB)
 Greatest lower bound = Infimum ($\cap$ LUB)

Lattices:-

A lattice L is a poset in which every pair of elements has a least upper bound LUB or supremum and a greatest lower bound GLB or infimum.

Join Consider a poset L under the ordering $\leq$
 Let $a, b \in L$ Then $LUB(a, b)$ or

Sup (a, b) is denoted by $a \vee b$ or $a \cup b$ and is called the join of a and b

$$\boxed{a \vee b = \sup(a, b)}$$

LUB.

Meet

Consider a poset L under the ordering $\leq$.
Let $a, b \in L$. Then $\inf(a, b)$ or $\inf(a, b)$ is denoted by $a \wedge b$ or $a \cap b$ and is called the meet of a & b

$$\boxed{a \wedge b = \inf(a, b)}$$

GLB.

From the above, it follows that a lattice L is a mathematical structure with two binary operations $\vee$ (join) and $\wedge$ (meet) denoted by $\{L, \vee, \wedge\}$. The lattice L for any element a, b, c satisfying the following properties.

I commutative property

(i) $a \wedge b = b \wedge a$

(ii) $a \vee b = b \vee a$

II Associative property

- (i) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
 (ii) $(a \vee b) \vee c = a \vee (b \vee c)$

III Absorption property

(1) $a \wedge (a \vee b) = a$

(2) $a \vee (a \wedge b) = a$

Exa: Consider the set $\{1, 2, 4, 8, 16\}$ with % as order. then it is a lattice.



$\{2, 4, 8\}$

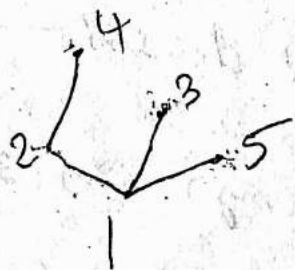
$LUB = 8, 16$

$least = \underline{8}$

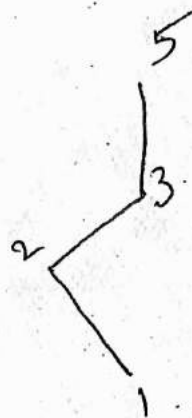
$\{2, 4\} =$

Exa: The set of rational nos with usual order is a lattice.

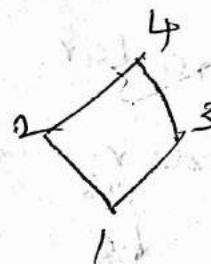
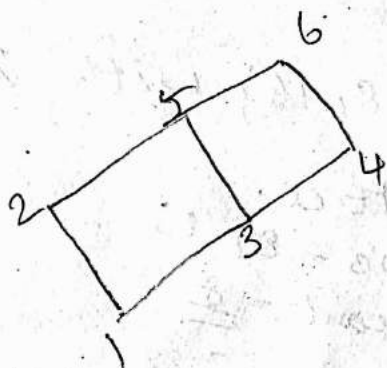
Exa: The poset $\{1, 2, 3, 4, 5\}$ whose Hasse-diagram is,



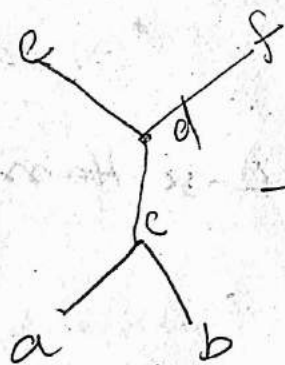
$\therefore 3, 5$ do not have join. So it is not a lattice.

Exac

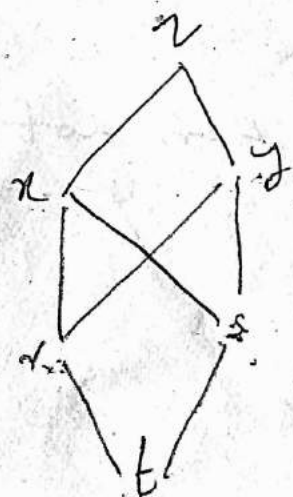
is a lattice. Any two elements have meet and join.



All are lattices.



→ not a lattice. The elements e, f have no upper bound $\therefore \sup(e, f)$ does not exist. Similarly a, b have no lower bounds $\inf(a, b)$ doesn't exist.



→ not a lattice. (x, y) has 3 lower bounds r, s, t : but $\inf(x, y) = \text{GLB}(x, y)$ doesn't exist. Also $\sup(x, y) = \text{LUB}(x, y)$ doesn't exist.

Result: Note:- In a lattice $(L, \leq)$

(R.1) 1. If $c \leq a, c \leq b$ then $c \leq a \wedge b$ $\left[\begin{array}{l} c \text{ is a lower bound of } a \text{ \& } b. \\ a \wedge b \text{ is the glb of } a \text{ \& } b. \end{array} \right]$

(R.2) 2. If $a \leq c, b \leq c$ then $a \vee b \leq c$ $\left[\begin{array}{l} \therefore c \text{ is an upper bound of } a, b. \\ a \vee b \text{ is the least upper bound of } a, b \end{array} \right]$

(R.3) 3. $a, b \leq a \vee b$ $\left[\begin{array}{l} \text{ie, } a \leq a \vee b \\ b \leq a \vee b. \end{array} \right]$

(R.4) 4. $a \wedge b \leq a, b$ $\left[\begin{array}{l} a \wedge b \leq a \\ a \wedge b \leq b. \end{array} \right]$

Property:-

If $(L, \leq)$ is a lattice. Then

(R.5) (1) $a \vee b = b$ iff $a \leq b$

(R.6) (2) $a \wedge b = a$ iff $a \leq b$

(R.7) (3) $a \wedge b = a$ iff $a \vee b = b$

Proof:- Let $a \vee b = b$

We have $a \leq a \vee b$ $\left\{ \begin{array}{l} \text{since } a \vee b = b \end{array} \right.$

ie, $a \leq b$

Let $a \leq b$. Then we have $b \leq a \vee b$

$a, b \leq b$ (by definition of a poset)
 $\therefore a \leq b \text{ \& } b \leq b$

$$\Rightarrow a \vee b \leq b.$$

$$\therefore \underline{a \vee b = b \text{ iff } a \leq b}$$

② Let $a \wedge b = a$.

$$\text{in, } \text{glb}\{a, b\} = a$$

$$\therefore \underline{a \leq b}$$

$$\text{Let } a \leq b.$$

$$\therefore a \wedge b \leq a. \text{ --- ①}$$

$$(a, a \wedge b(a, b) \leq a)$$

$$\text{now } a \leq b \text{ \& } a \leq a$$

$$\Rightarrow a \leq b \wedge a$$

$$\Rightarrow a \leq a \wedge b \text{ --- ②}$$

$$\text{from ① \& ②}$$

$$\underline{a \wedge b = a}$$

(iii) Combining (i) & (ii) we get,

$$a \wedge b = a \text{ iff } a \leq b \quad (\text{eqn (2)})$$

$$\text{ie, } \underline{a \wedge b = a \text{ iff } a \vee b = b} \quad (\text{from eqn (1)})$$

Isotonic properties

If $(L, \leq)$ is a lattice. Then for any $a, b, c \in L$ the following properties hold good.

(Q.8) If $b \leq c$ then

(i) $a \vee b \leq a \vee c$
(ii) $a \wedge b \leq a \wedge c$

Proof: Given $b \leq c$

$$\therefore b \leq c \leq a \vee c \quad \text{--- (1)}$$

$$\text{Also we have } a \leq a \vee c \quad \text{--- (2)}$$

$$\text{ie, } a \leq a \vee c \text{ \& } b \leq a \vee c$$

$$\therefore \underline{a \vee b \leq a \vee c}$$

(ii) $a \wedge b \leq b \leq c$

$$a \wedge b \leq a$$

$$\therefore a \wedge b \leq a \text{ \& } a \wedge b \leq c$$

$$\Rightarrow \underline{a \wedge b \leq a \wedge c}$$

Distributive inequalities

If $(L, \leq)$ is a lattice, then for any $a, b, c \in L$

$$\begin{aligned} \text{(i)} \quad a \wedge (b \vee c) &\geq (a \wedge b) \vee (a \wedge c) \\ \text{(ii)} \quad a \vee (b \wedge c) &\leq (a \vee b) \wedge (a \vee c) \end{aligned}$$

Proof: we know, $a \wedge b \leq a$ — (1)

$$a \wedge b \leq b \leq b \vee c \quad \text{--- (2)}$$

from (1) & (2) we get

From (R.I)

$$a \wedge b \leq a \wedge (b \vee c) \quad \text{--- (3)}$$

$$a \wedge c \leq a$$

$$a \wedge c \leq c \leq b \vee c$$

$$\therefore a \wedge c \leq a \wedge (b \vee c) \quad \text{(From R.I)} \quad \text{--- (4)}$$

$$\text{from (3) \& (4) } \underline{\underline{(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)}}$$

$$\text{ii, } \underline{\underline{a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)}}$$

(2)

$$a \leq a \vee b$$

$$b \wedge c \leq b \leq a \vee b$$

$$\text{ii, } b \wedge c \leq a \vee b$$

ie, $a \leq a \vee b$ & $b \wedge c \leq a \vee b$ then

$$a \vee (b \wedge c) \leq a \vee b \quad \text{--- (1)}$$

Now $a \leq a \vee c$

$$b \wedge c \leq c \leq a \vee c$$

ie, $a \leq a \vee c$ & $b \wedge c \leq a \vee c$

$\vee \rightarrow \text{LUB}$

$\wedge \rightarrow \text{GLB}$

$$\therefore a \vee (b \wedge c) \leq a \vee c \quad \text{--- (2)}$$

$$\therefore \underline{a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)}$$

Modular Inequality

If L is a lattice then, $\boxed{a \leq c \overset{\text{iff}}{\iff} a \vee (b \wedge c) \leq (a \vee b) \wedge c}$

Proof: — Let $a \leq c$

We know, $\boxed{a \vee c = c}$ (by previous theorem. $a \vee b = b$ iff $a \leq b$)

By distributive property,

$$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$$

$$\text{ie, } \underline{a \vee (b \wedge c) \leq (a \vee b) \wedge c} \quad \text{from (1)}$$

conversely let $a \vee (b \wedge c) \leq (a \vee b) \wedge c$

we have

$$a \leq a \vee (b \wedge c)$$

and $(a \vee b) \wedge c \leq c$

$$\text{ie, } a \leq a \vee (b \wedge c) \leq (a \vee b) \wedge c \leq c$$

$$\Rightarrow \underline{a \leq c}$$

Q. If $P(S)$ is the powerset of S and $\cup, \cap$ are taken as the join and meet then P.T, $\{P(S), \subseteq\}$ is a lattice.

Proof: Let A & B be any two elements of $P(S)$ i.e. any 2 subsets of S .

Then an upper bound of $\{A, B\}$ is a subset of S that contains both A and B and the least among them is $A \cup B$.

clearly $A \cup B \in P(S)$.

We know $A \subseteq A \cup B$ & $B \subseteq A \cup B$

$\therefore A \cup B$ is an upper bound of $\{A, B\}$

Also $A \cup B$ is the least upper bound of (A, B)

Similarly $A \cap B \subseteq A$ and $A \cap B \subseteq B$

ie, $A \cap B$ is the lower bound of $\{A, B\}$

Now clearly $A \cap B$ is greatest lower bound of $\{A, B\}$, since if there exist another lower bound C for A & B . Then,

$$C \subseteq A \quad \text{and} \quad C \subseteq B$$

$$\Rightarrow C \subseteq A \cap B.$$

$$\therefore \text{glb}(A, B) = A \cap B$$

$\therefore \{P(S) \subseteq\}$ is a lattice

Sub Lattices

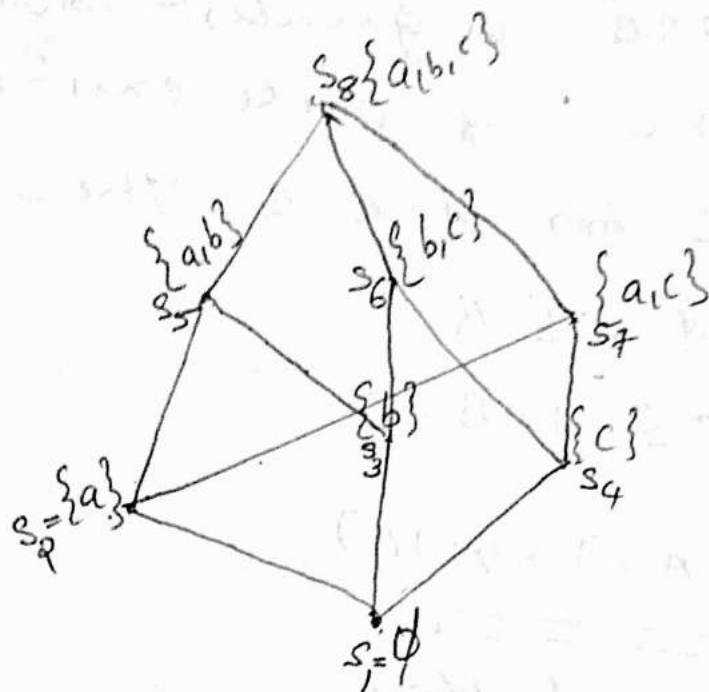
A non empty subset M of a lattice $\{L, \vee, \wedge\}$ is called a sublattice of L iff M is closed under both operations $\vee, \wedge$
ie, $a, b \in M$ then $a \vee b \in M$ & $a \wedge b \in M$

Eg: Let $S = \{a, b, c\}$ consider $(P(S), \subseteq)$ we have seen this as a lattice. Examine

$$\text{Whether } L_1 = \{S_1, S_2, S_3, S_6\} \quad L_2 = \{S_1, S_5, S_7, S_8\}$$

$$L_3 = \{s_1, s_4, s_6, s_8\} \quad \& \quad L_4 = \{s_1, s_2, s_4, s_7\}$$

are sub lattices of $P(S)$.



(1) consider $L_1 = \{s_1, s_2, s_4, s_6\}$

$$s_1 \vee s_2 = s_2 \quad s_2 \vee s_4 = s_7$$

$$s_1 \vee s_4 = s_4 \quad s_2 \vee s_6 = s_8$$

$$s_1 \vee s_6 = s_6$$

Thus $s_2 \vee s_4 = s_7$ but $s_7 \notin L_1$.

From this we can say L_1 is not a sub lattice.

(2) $L_2 = \{s_1, s_5, s_7, s_8\}$

$$s_1 \vee s_5 = s_5$$

$$s_1 \vee s_8 = s_8$$

$$s_5 \vee s_8 = s_8$$

$$s_1 \vee s_7 = s_7$$

$$s_5 \vee s_7 = s_8$$

$$s_7 \vee s_8 = s_8$$

$$s_1 \wedge s_5 = s_1 \quad s_5 \wedge s_7 = s_2$$

$$s_1 \wedge s_7 = s_1$$

$$s_1 \wedge s_8 = s_1$$

but $s_2 \notin L_2 \quad \therefore L_2$ is not a sublattice

$$(3) \quad L_3 = \{s_2, s_4, s_6, s_8\}$$

$$s_2 \vee s_4 = s_7 \quad s_4 \vee s_6 =$$

$$s_2 \vee s_6 = s_8$$

$$s_2 \vee s_8 =$$

clearly $s_2 \vee s_4 = s_7 \notin L_3 \quad \therefore L_3$ is not a sublattice

$$(4) \quad L_4 = \{s_1, s_2, s_4, s_7\}$$

$$s_1 \vee s_2 = s_2 \quad s_2 \vee s_4 = s_7$$

$$s_1 \vee s_4 = s_4 \quad s_2 \vee s_7 = s_7$$

$$s_1 \vee s_7 = s_7 \quad s_4 \vee s_7 = s_7$$

$$s_1 \wedge s_2 = s_1$$

$$s_2 \wedge s_7 = s_1$$

$$s_1 \wedge s_4 = s_1$$

$$s_2 \wedge s_7 = s_2$$

$$s_1 \wedge s_7 = s_1$$

$$s_4 \wedge s_7 = s_4$$

$\therefore L_4$ is a sublattice

Exa: If $a, b \in L$ such that $a \leq b$ and
 If the interval $[a, b]$ is defined as the
 set of all $x \in L$, $a \leq x \leq b$ s.t. $[a, b]$

is a sub lattice of L .

Pr: s.t. every interval of a lattice is a sublattice.

Let $x, y \in [a, b]$ Then we know,

$$x, y \in L.$$

Also $x \vee y \in L$ and $x \wedge y \in L$

Since L is a lattice.

Now, we have $a \leq x \leq b$

but $x \leq x \vee y$ and

$$x \wedge y \leq x.$$

$$\therefore a \leq x \leq x \vee y \leq b$$

$$a \leq x \wedge y \leq x \leq b$$

Hence $x \vee y \in [a, b]$

$$x \wedge y \in [a, b]$$

$\therefore [a, b]$ is a sublattice

Distributive lattice

A Lattice $\{L, \vee, \wedge\}$ is called a distributive lattice if for any elements $a, b, c \in L$

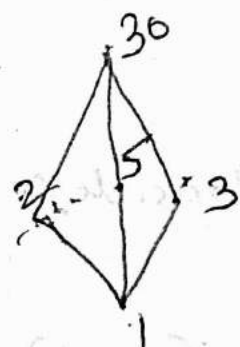
$$\begin{aligned} a \wedge (b \vee c) &= (a \wedge b) \vee (a \wedge c) \\ \text{and} \quad a \vee (b \wedge c) &= (a \vee b) \wedge (a \vee c) \end{aligned}$$

If the lattice L does not satisfy the above properties, it is called a non-distributive lattice.

Ex: consider $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$.

Now $L = \{1, 2, 3, 5, 30\}$ is a sublattice of D_{30} .
Check whether it is a distributive lattice.

LUB V	1	2	3	5	30
1	1	2	3	5	30
2	2	2	30	30	30
3	3	30	3	30	30
5	5	30	30	5	30
30	30	30	30	30	30



LUB Λ	1	2	3	5	30
1	1	1	1	1	1
2	1	2	1	1	2
3	1	1	3	1	3
5	1	1	1	5	5
30	1	2	3	5	30

$$\text{Now, } 2 \vee (5 \wedge 3) = 2 \vee 1 = 2$$

$$(2 \vee 5) \wedge (2 \vee 3) = 30 \wedge 30 = 30$$

$$\therefore 2 \vee (5 \wedge 3) \neq (2 \vee 5) \wedge (2 \vee 3)$$

$\therefore L$ is not a distributive lattice

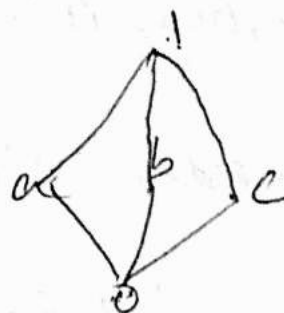
H.W

Q. P.T. the given lattice is not distributive

Proof

$$0 \leq a \quad 0 \leq b \quad 0 \leq c$$

$$a \leq 1 \quad b \leq 1 \quad c \leq 1$$



Also

$$a \wedge b = 0$$

$$a \wedge c = 0$$

$$b \wedge c = 0$$

$$a \vee b = 1$$

$$a \vee c = 1$$

$$b \vee c = 1$$

Consider

$$a \wedge (b \vee c) = a \wedge 1 = \underline{\underline{a}}$$

$$(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$$

$$\therefore a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$$

$\therefore L$ is not a distributive lattice.

H.W

Q. Let $(L, \vee, \wedge)$ be a distributive lattice under ' $\leq$ '. For any $a, b, c \in L$. If $a \wedge b = a \wedge c$

$a \vee b = a \vee c$ then p.T $b = c$

Proof:- For a distributive lattice,

$$(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$$

$$\text{L.H.S} \rightarrow (a \wedge b) \vee c = (a \wedge c) \vee c$$

$$= \underline{\underline{c}}$$

Since $(a \wedge c) \leq c$

$$\text{R.H.S} \Rightarrow (a \vee c) \wedge (b \vee c) = (a \vee b) \wedge (b \vee c)$$

$$= (b \vee a) \wedge (b \vee c)$$

$$\Rightarrow b \vee (a \wedge c) \quad (\text{distributive})$$

$$\Rightarrow b \vee (a \wedge b) \quad \text{given condition}$$

$$\Rightarrow \underline{\underline{b}}$$

since $a \wedge b \leq b$.

For a distributive lattice

$$\text{i.e. } \underline{\underline{b = c}}$$

Bounded Lattice

A Lattice L is called a bounded lattice if it has a greatest element 1 and a least element 0 .

Exa:- The powerset $P(S)$ of the set S under the operations of intersection and union is a bounded lattice since ϕ is the least element of $P(S)$ and the set S is the greatest element of $P(S)$.

- (ii) The set of +ve integers under usual order of $\leq$ is not a bounded lattice since it has a least element 1, but the greatest element doesn't exist.

Properties of Bounded lattice

If L is a bounded lattice, then for any element $a \in L$, we have the following results:-

$$(i) \quad a \vee 1 = 1$$

$$(ii) \quad a \wedge 1 = a$$

$$(ii') \quad a \vee 0 = a$$

$$(iv) \quad a \wedge 0 = 0.$$

Q. Prove that every finite lattice is $L = \{a_1, a_2, \dots, a_n\}$ is bounded.

Let $L = \{a_1, a_2, a_3, \dots, a_n\}$ be a lattice.

Thus the greatest element of Lattice is

$$a_1 \vee a_2 \vee a_3 \vee a_4 \vee \dots \vee a_n$$

and the least element of lattice is

$$a_1 \wedge a_2 \wedge a_3 \wedge \dots \wedge a_n$$

Since the greatest and least elements

exists for every finite lattice. Therefore L is bounded.

⑧ S.T the lattices shown below are non distributive.

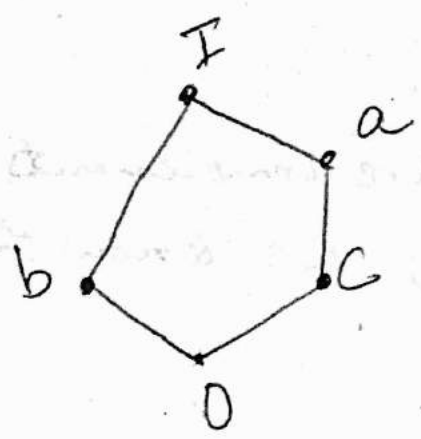


fig. 1

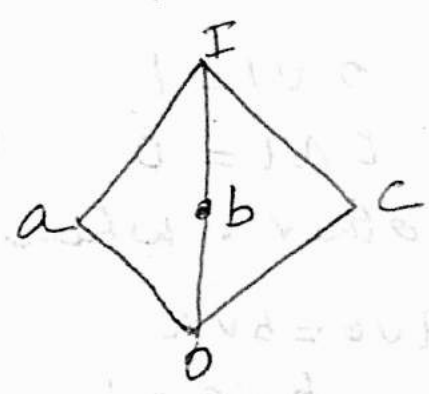


fig. 2

From fig 1

$$a \wedge (b \vee c) = a \wedge I = \underline{\underline{a}}$$

$$\text{but } (a \wedge b) \vee (a \wedge c) = 0 \vee c = c$$

$$\therefore a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$$

$\therefore$ the lattice is not distributive.

from fig 2 $a \wedge (b \vee c) = a \wedge I = a$

$$(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$$

$\therefore$ Here also lattice is not distributive

Complemented Lattice

If L is a bounded lattice and $a \in L$. Then Con
an element $b \in L$ is called a complement
of a if

$$\begin{aligned} a \vee b &= 1 \\ a \wedge b &= 0. \end{aligned}$$

Since $0 \vee 1 = 1$

$0 \wedge 1 = 0$ 0 and 1 are complements of
each other. When $a \vee b = 1$, we know that

$$a \vee b = b \vee a$$

$$\therefore \underline{b \vee a = 1}$$

$$\text{When } a \wedge b = 0 \Rightarrow b \wedge a = 0$$

Hence when b is the complement of a , a
is the complement of b .

Note:-

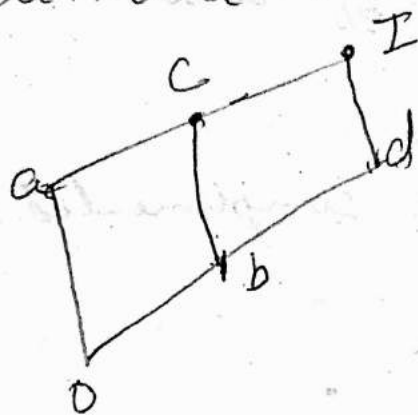
It is not necessary that an element x has
a complement. Also an element have
more than one complement.

$$1' = 0 \quad \text{and} \quad 0' = 1$$

Defn:-

A lattice L is called a complemented lattice
if L is bounded and every element in L
has a complement.

Qna:- Determine the complement of a and c

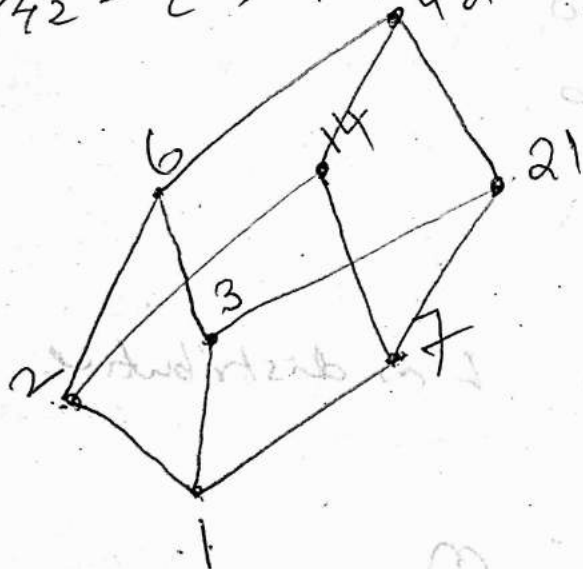


Complement of a is d since $a \vee d = 1$
 $a \wedge d = \underline{\underline{0}}$

Complement of c doesn't exist. There doesn't exist a' s. $c \vee c' = 1$
 $c \wedge c' = \underline{\underline{0}}$

⑧ S-T D_{42} with ' $'$ ' as order is a complemented lattice.

$$D_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$$



$$1' = 42$$

$$2' = 21$$

$$3' = 14$$

$$6' = 7$$

$$7' = 6$$

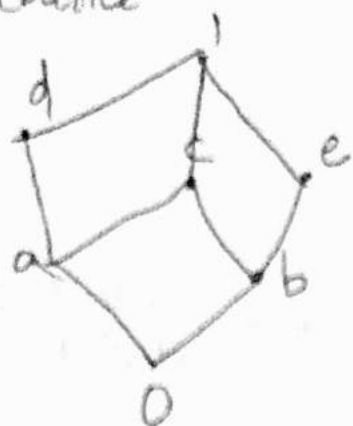
$$14' = 3$$

$$21' = 2$$

every element possess its complement. Therefore it is a complemented lattice.

H.W. (1) Examine whether D_{30} , D_{50} are complemented lattice

(2)



is it a complemented lattice

Ans: c doesn't have complement.

(3) In a distributive lattice if an element $a \in L$ has a complement then it is unique.

Let $a \in L$ assume that a has two complements b and c . Then,

$$a \vee b = 1 \quad a \wedge b = 0$$

$$a \vee c = 1 \quad a \wedge c = 0$$

Now,

$$b = b \vee 0$$

$$= b \vee (a \wedge c)$$

$$= (b \vee a) \wedge (b \vee c) \quad L \text{ is distributive}$$

$$= 1 \wedge (b \vee c)$$

$$= \underline{b \vee c} \quad \text{--- (1)}$$

$$c = c \vee 0$$

$$= c \vee (a \wedge b)$$

$$= (c \vee a) \wedge (c \vee b) \quad \text{---}$$

$$= 1 \wedge (c \vee b)$$

$$= c \vee b$$

$$= \underline{b \vee c} \quad \text{--- ②.}$$

Thus from ① & ② $\Rightarrow b = c$. ie, the complements of a are the same. it is unique.

Boolean Algebra

If a lattice is complement and distributive then it is a boolean Algebra. A boolean Algebra is a lattice which contains a least element and a greatest element and which is both complemented and distributive.

Note: In a boolean Algebra complements of an element a is unique.

IInd Defn of Boolean Algebra

If B is a non empty set with 2 binary operations $+$ and $\cdot$, two distinct elements 0 and 1 , then B is called a boolean Algebra.

Basic boolean Algebra laws

$$\text{I} \quad \left. \begin{array}{l} a + 0 = a \\ a \cdot 1 = a \end{array} \right\} \text{identity laws}$$

Module IVRecurrence relation

Defn: An equation that expresses a_n i.e., the general term of the sequence $\{a_n\}$ in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, a_2, \dots, a_{n-1}$ for all integers n with $n \geq n_0$ where n_0 is a non negative integer is called a recurrence relation for $\{a_n\}$ or a difference equation.

A recurrence relation of the form $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k} = f(n)$ is called a linear recurrence relation of degree k with const coefficients. $C_0, C_1, C_2, \dots, C_k$ are real numbers and $C_k \neq 0$.

The recurrence relation is called linear because each a_r is raised to the power 1.

The degree is the difference between the greatest and least subscripts of the members of sequence occurring in the recurrence relation.

If $f(n) = 0$ the recurrence relation is said to be homogeneous. Otherwise it is said to be non homogeneous.

Solving the recurrence relation

Let us consider the soln of a homogeneous recurrence relation of degree two, which is of the form is,

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0 \quad \forall n \geq 2 \quad \text{--- (1)}$$

Let $a_n = r^n$ be the solution of (1) then

$$C_0 r^n + C_1 r^{n-1} + C_2 r^{n-2} = 0$$

$$\text{or } r^{n-2} [C_0 r^2 + C_1 r + C_2] = 0$$

which is a quadratic equation in r and it is called auxiliary eqⁿ whose roots are r_1 and r_2 . These roots are called characteristic roots of the recurrence relation.

There are 3 cases: -

Case I Distinct real roots say r_1 and r_2

then C.F is,

$$a_n = C_1 r_1^n + C_2 r_2^n$$

Case 2 Equal roots $r_1 = r_2$ then

C.F is,

$$a_n = (C_1 + n C_2) r^n$$

Case 3: Complex roots say $r_1 = a + ib$ and $r_2 = a - ib$

$$r_1 = a + ib = r_1 (\cos \theta + i \sin \theta)$$

$$r_2 = a - ib = r_2 (\cos \theta - i \sin \theta)$$

$\therefore a_n = r^n [C_1 \cos n\theta + i C_2 \sin n\theta]$ where C_1 and C_2 are complex constants.

Q8) Solve $a_{n+2} + 3a_{n+1} + 2a_n = 0$. — (1)

Let $a_n = r^n$ be the soln of (1) then

$$r^{n+2} + 3r^{n+1} + 2r^n = 0$$

A.E is,

$$r^n (r^2 + 3r + 2) = 0 \Rightarrow r^2 + 3r + 2 = 0$$

$$(r+1)(r+2) = 0$$

$\Rightarrow r = -1, -2$ real & distinct roots

$$\therefore a_n = C_1 (-1)^n + C_2 (-2)^n$$

Q9) Solve $a_{n+2} + 2a_{n+1} + a_n = 0$

Let $a_n = r^n$ be the soln then

$$r^{n+2} + 2r^{n+1} + r^n = 0$$

$$r^n(r^2 + 2r + 1) = 0$$

$$\Rightarrow (r+1)^2 = 0 \Rightarrow r = -1, -1$$

equal roots.

$$\therefore \text{soln is, } \underline{a_n = (c_1 + n c_2)(-1)^n}$$

(8) solve $a_{n+2} + a_{n+1} + a_n = 0$.

let $a_n = r^n$ be the soln. $\therefore r^{n+2} + r^{n+1} + r^n = 0$

$$r^n(r^2 + r + 1) = 0 \Rightarrow r^2 + r + 1 = 0$$

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

$$\frac{-1 \pm \sqrt{3}i}{2} = r(\cos \theta \pm i \sin \theta)$$

$$r = \sqrt{a^2 + b^2} = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \underline{\underline{1}}$$

$$r \cos \theta = \frac{-1}{2} \quad r \sin \theta = \frac{\sqrt{3}}{2}$$

$$\cos \theta = \frac{-1}{2} \quad \sin \theta = \frac{\sqrt{3}}{2}$$

$$\therefore \theta = \frac{2\pi}{3} \text{ in 2nd quadrant.}$$

$$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\therefore a_n = 1^n \left(C_1 \cos \frac{2n\pi}{3} + C_2 \sin \frac{2n\pi}{3} \right)$$

$$\underline{\underline{a_n = C_1 \cos \left(\frac{2n\pi}{3} \right) + C_2 \sin \left(\frac{2n\pi}{3} \right)}}$$

⑧ $a_{n+2} - \sqrt{3} a_{n+1} + a_n = 0$

A.E.M, $r^2 - \sqrt{3}r + 1 = 0$

$$r = \frac{\sqrt{3} \pm \sqrt{3-4}}{2} = \frac{\sqrt{3} \pm i}{2}$$

$$r = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \underline{\underline{1}}$$

$$\frac{2n\pi}{3}$$

$$r \cos \theta = \frac{\sqrt{3}}{2}$$

$$r \sin \theta = \frac{+1}{2}$$

$$\cos \theta = \frac{\sqrt{3}}{2}$$

$$\sin \theta = \frac{+1}{2}$$

$$\therefore \theta = \frac{\pi}{6}$$

$$\therefore a_n = \underline{\underline{C_1 \cos \left(\frac{n\pi}{6} \right) + C_2 \sin \left(\frac{n\pi}{6} \right)}}$$

⑨ $a_{n+2} + 4a_{n+1} + 4a_n = 0$

A.E.M, $r^2 + 4r + 4 = 0$

$$\Rightarrow (r+2)^2 = 0 \quad r = -2, -2 \text{ equal roots}$$

$$a_n = (q + nr) \cdot f(r)^n$$

Non homogeneous Recurrence relation

The solution of a linear non homogeneous recurrence relation with constant coefficients, is a recurrence relation of the form

$$c_0 a_n + c_1 a_{n-1} + \dots + c_k a_{n-k} = f(n) \quad \text{where}$$

$$f(n) \neq 0 \text{ is of the form, } a_n = a_n^{(h)} + a_n^{(p)}$$

where $a_n^{(h)}$ is the solution of the corresponding homogeneous recurrence relation, namely $c_0 a_n + c_1 a_{n-1} + \dots + c_k a_{n-k} = 0$

$a_n^{(p)}$ is the particular soln of the above non homogeneous recurrence relation

form of $f(n)$	Form of $a_n^{(p)}$
$c_1 \text{ const}$	$A, \text{ constant}$
r^n	$C r^n$
n	$A_0 + A_1 n$
n^2	$A_0 + A_1 n + A_2 n^2$
a polynomial of degree n	$(A_0 r^n + A_1 r^{n-1} + \dots + A_n)$

$n^t, t \in \mathbb{Z}$	$A_0 + A_1 n + A_2 n^2 + \dots + A_k n^k$
$r^n n^t$	$r^n (A_0 + A_1 n + A_2 n^2 + \dots + A_k n^k)$
$\sin n\alpha$	$C_1 \cos n\alpha + C_2 \sin n\alpha$
$\cos n\alpha$	$C_1 \cos n\alpha + C_2 \sin n\alpha$
$r^n \cos n\alpha$	$r^n (C_1 \cos n\alpha + C_2 \sin n\alpha)$
$r^n \sin n\alpha$	$r^n (C_1 \cos n\alpha + C_2 \sin n\alpha)$

(18) Solve the recurrence relation, $a_{n+1} - 2a_n = 5$ with $a_0 = 1$

Soln is $a_n^{(p)} + a_n^{(h)}$

homo. eqⁿ is, $a_{n+1} - 2a_n = 0$.

let $a_n = r^n$ be the soln,

$$r^{n+1} - 2r^n = 0 \Rightarrow r^n (r - 2) = 0$$

$$\Rightarrow \underline{\underline{r = 2}}$$

$$\therefore \underline{\underline{a_n^{(h)} = C \cdot 2^n}}$$

Corres. non homogeneous

Let $a_n^{(P)} = A$ then be the soln of given.

$$\therefore \textcircled{1} \Rightarrow A - 2A = 5$$

$$-A = 5 \quad \therefore A = -5$$

$$\therefore a_n^{(P)} = -5$$

$$\therefore \text{solution is, } a_n = a_n^{(h)} + a_n^{(P)}$$

$$a_n = C_1 2^n - 5$$

given $a_0 = 1$ in, $1 = C_1 - 5$

$$\therefore C_1 = \underline{\underline{6}}$$

$$\therefore \underline{\underline{a_n = 6 \cdot 2^n - 5}} \text{ is the soln of } \textcircled{1}$$

(18) Solve the recurrence relation,
 $a_{n+2} - 3a_{n+1} + 2a_n = 3^n \quad a_0 = 3, a_1 = 6$

Soln is of the form $a_n = a_n^{(h)} + a_n^{(P)}$

homog. eqn is, $a_{n+2} - 3a_{n+1} + 2a_n = 0.$

A.E is,

$$r^2 - 3r + 2 = 0$$

$$(r-1)(r-2) = 0 \Rightarrow r = 1, 2$$

$$\therefore a_n^{(h)} = C_1 1^n + C_2 2^n \quad \therefore a_n^{(h)} = C_1 + C_2 2^n$$

Corr. Non homogeneous soln is $a_n^{(P)}$

here $f(n) = 3^n$ we choose $a_n = C 3^n$

$\therefore$ let $a_n = C 3^n$ be the soln of given eqⁿ.

Then the eqⁿ becomes,

$$C 3^{n+2} - 3C 3^{n+1} + 2C 3^n = 3^n$$

$$C 3^n (3^2 - 3^2 + 2) = 3^n$$

$$2C 3^n = 3^n$$

$$C = \frac{1}{2}$$

$$\therefore a_n^{(P)} = \frac{1}{2} 3^n$$

$\therefore$ general soln is, $a_n = a_n^{(h)} + a_n^{(P)}$

$$a_n = C_1 + C_2 2^n + \frac{3^n}{2}$$

given $a_0 = 3$

$$\text{in, } a_0 = C_1 + C_2 + \frac{1}{2}$$

$$3 = C_1 + C_2 + \frac{1}{2}$$

$$C_1 + C_2 = \frac{5}{2} \quad \text{--- (1)}$$

given $a_1 = 6$

$$\Rightarrow a_1 = C_1 + C_2 2 + \frac{3}{2}$$

$$6 = C_1 + 2C_2 + \frac{3}{2}$$

Solving
$$\begin{cases} C_1 + 2C_2 = \frac{9}{2} & \text{--- (2)} \\ C_1 + C_2 = \frac{5}{2} & \text{--- (1)} \end{cases}$$

$$C_2 = \underline{\underline{2}}$$

$$C_1 + 2 = \frac{5}{2}$$

$$C_1 = \frac{5}{2} - 2 = \underline{\underline{\frac{1}{2}}}$$

Hence particular soln is,

$$a_n = \underline{\underline{\frac{1}{2} + 2 \cdot 2^n + \frac{3^n}{2}}}$$

⑧ Solve $a_n - 3a_{n-1} = 3^n$, $n \geq 1$, $a_0 = 1$

Homog. eqn is, $a_n - 3a_{n-1} = 0$.

Let $a_n = r^n$ be the soln. then

$$r^n - 3r^{n-1} = 0$$

$$r^{n-1}(r-3) = 0 \Rightarrow r = \underline{\underline{3}}$$

$$\therefore a_n^{(h)} = \underline{\underline{C_1 3^n}}$$

Corresponding Non homogeneous equation

$$\text{is } a_n - 3a_{n-1} = 3^n$$

Here $f(n) = 3^n$. But 3^n is already a soln.

$\therefore$ we take $a_n = nC_1 3^n$ is another soln.

Substituting then we get,

$$nC3^n - 3(n-1)C3^{n-1} = 3^n$$

$$C3^{n-1} [3n - 3(n-1)] = 3^n$$

$$C3^{n-1} (3n - 3n + 3) = 3^n$$

$$3C3^{n-1} = 3^n$$

$$\Rightarrow C = \underline{\underline{1}}$$

$$A_n^{(P)} = \underline{\underline{n3^n}}$$

$$\therefore \text{general soln is, } A_n = C_1 3^n + n3^n$$

$$\therefore A_{n-1} = C_1 3^{n-1} + (n-1)3^{n-1}$$

$$\text{Given } A_0 = 1, \text{ or } 1$$

$$\Rightarrow \text{put } n=1 \Rightarrow 1 = \underline{\underline{C_1}}$$

$$\therefore \text{particular soln is, } A_n = 3^n + n3^n$$

$$A_n = \underline{\underline{3^n(1+n)}}$$

Q) Solve $A_{n+2} - 3A_{n+1} + 2A_n = 2^n$, $A_0 = 3$, $A_1 = 6$

Homo. eqn is, $A_{n+2} - 3A_{n+1} + 2A_n = 0$

$$r^{n+2} - 3r^{n+1} + 2r^n = 0$$

$$\Rightarrow r^n (r^2 - 3r + 2) = 0$$

$$\Rightarrow (r-1)(r-2) = 0$$

$$\Rightarrow r = 1, 2$$

$$\therefore \text{Soln is, } a_n = c_1 1^n + c_2 2^n = \underline{\underline{c_1 + c_2 2^n}}$$

Corresponding Non homogeneous equation

$$\text{is, } a_{n+2} - 3a_{n+1} + 2a_n = 2^n$$

Here $f(n) = 2^n$ but 2^n is one of the

Soln of homo. eqn.

$$\therefore \text{we take } a_n = nC 2^n$$

$$(n+2)C 2^{n+2} - 3(n+1)C 2^{n+1} + 2nC 2^n = 2^n$$

$$C \cdot 2^n [(n+2) \cdot 2^2 - 3(n+1) \cdot 2 + 2n] = 2^n$$

$$\Rightarrow C 2^n [4n + 8 - 6n - 6 + 2n] = 2^n$$

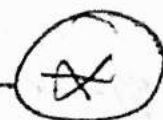
$$\Rightarrow C [2] = 1 \Rightarrow C = \frac{1}{2}$$

$$\therefore a_n^{(P)} = n \cdot \frac{1}{2} \cdot 2^n = \underline{\underline{n \cdot 2^{n-1}}}$$

general soln is,

$$a_n = a_n^{(h)} + a_n^{(P)}$$

$$a_n = c_1 + c_2 2^n + n 2^{n-1}$$



Given $a_0 = 3$ and $a_1 = 6$.

Sub. these in $(*)$.

$$a_0 = c_1 + c_2 = 3 \quad \text{and}$$

$$a_1 = c_1 + 2c_2 + 1 = 6$$

$$c_1 + 2c_2 = 5$$

Solving

$$c_1 + c_2 = 5$$

$$c_1 + c_2 = 3$$

$$c_2 = 2$$

$$\therefore c_1 = 1$$

$\therefore$ general soln is,

$$a_n = 1 + 2^{n+1} + n2^{n-1}$$

Que:-

(Q) Solve $a_n = 2a_{n-1} + 3 \cdot 2^n$, $a_0 = 5$

Given $a_n - 2a_{n-1} = 3 \cdot 2^n$

Corr. homo. eqn is, $a_n - 2a_{n-1} = 0$

$$r^n - 2r^{n-1} = 0$$

$$r^{n-1}(r - 2) = 0 \Rightarrow r = \underline{2}$$

Soln is $a_n^{(h)} = c_1 2^n$

Corr. non homo. eqn is, $a_n - 2a_{n-1} = 3 \cdot 2^n$

Here $f(n) = 3 \cdot 2^n$ but 2^n is the soln of homo. eqn. $\therefore$ we take $a_n = nC 2^n$.

$$nC2^n - 2(n-1)C2^{n-1} = 3 \cdot 2^n$$

$$C2^{n-1}[2n - 2n + 2] = 3 \cdot 2^n$$

$$2C = 3 \cdot 2$$

$$\Rightarrow C = \underline{\underline{3}}$$

$$\therefore \underline{\underline{a_n^{(P)} = 3n \cdot 2^n}}$$

general soln is, $a_n^{(h)} + a_n^{(P)} \Rightarrow C_1 2^n + 3n \cdot 2^n$

Given $a_0 = 5$

$$a_n = C_1 2^n + 3n \cdot 2^n$$

$$\xrightarrow{n=0}$$

$$a_0 = C_1 2^0 + 0$$

$$5 = \underline{\underline{C_1}}$$

$\therefore$ Soln is, $a_n = 5 \cdot 2^n + 3n \cdot 2^n$

$$\underline{\underline{a_n = (5 + 3n) 2^n}}$$

H.W. (8) Solve $9a_n = 6a_{n-1} - a_{n-2}$, $a_0 = 3$, $a_1 = -1$

hint: (homo)

$$9a_n - 6a_{n-1} + a_{n-2} = 0$$

$$a_n = (3 - 6n) \left(\frac{1}{3}\right)^n$$

Q Solve $a_n - 2a_{n-1} = n+5$, $n \geq 1$, $a_0 = 4$

Corr. homo eqⁿ is $a_n - 2a_{n-1} = 0$

Let $a_n = r^n$ be the soln

$$r^n - 2r^{n-1} = 0 \Rightarrow r^{n-1}(r-2) = 0.$$

$$\Rightarrow r = \underline{\underline{2}}$$

$\therefore$ soln is, $a_n = \underline{\underline{C \cdot 2^n}}$

$f(n) = n+5$ then we choose the soln,

$$a_n^{(P)} = A + Bn$$

$$\therefore A + Bn - 2(A + B(n-1)) = n+5$$

$$A + Bn - 2A - 2Bn + 2B = n+5$$

$$-A - Bn + 2B = n+5$$

equating co. of n both sides.

$$-B = 1 \quad \therefore B = \underline{\underline{-1}}$$

$$-A + 2B = 5$$

$$-A - 2 = 5$$

$$-A = \underline{\underline{7}} \quad \therefore A = \underline{\underline{-7}}$$

$$\therefore a_n^{(P)} = -7 - n$$

$\therefore$ general soln is, $a_n = a_n^{(h)} + a_n^{(p)}$

$$\underline{\underline{a_n = C_1 2^n - 7 - n}} \quad \text{--- (1)}$$

Given $a_0 = 4 \quad n \geq 1$

$\therefore$ from (1) we can write

$$a_{n-1} = C_1 2^{n-1} - 7 - (n-1)$$

put $n=1 \Rightarrow a_0 = C_1 2^0 - 7 - 0$

$$a_0 = C_1 - 7$$

$$4 = C_1 - 7$$

$$\therefore C_1 = \underline{\underline{11}}$$

$\therefore$ particular soln is, $\underline{\underline{a_n = 11 \cdot 2^n - (n+7)}}$

(Q) Solve $a_n - 2a_{n-1} = 2n^2, \quad n \geq 1 \quad a_1 = 4.$

Corr. homo. eqn is, $a_n - 2a_{n-1} = 0.$

Let $a_n = r^n$ be the soln.

$$r^n - 2r^{n-1} = 0.$$

$$r^{n-1}(r-2) = 0 \Rightarrow r = 2.$$

Soln is, $\underline{\underline{a_n^{(h)} = C_1 2^n}}$

$f(n) = 2n^2 \therefore$ we take $a_n^{(P)} = An^2 + Bn + C$.

$$(An^2 + Bn + C) - 2(A(n-1)^2 + B(n-1) + C) = 2n^2$$

$$An^2 + Bn + C - 2[A(n^2 - 2n + 1) + Bn - B + C] = 2n^2$$

$$An^2 - 2An^2 + Bn - \cancel{4}An + C - 2A - 2Bn + 2B - 2C = 2n^2$$

equating Co. of n^2

$$A - 2A = 2$$

$$-A = 2 \therefore A = \underline{\underline{-2}}$$

coeff. of n

$$B - \cancel{4}A - 2B = 0$$

$$-B - \cancel{4}A = 0$$

$$-B + 8 = 0 \therefore B = \underline{\underline{-8}}$$

const term

$$C - 2A + 2B - 2C = 0$$

$$-C + 4 - 16 = 0$$

$$-C - 12 = 0$$

$$C = \underline{\underline{-12}}$$

$$a_n^{(P)} = \underline{\underline{-2n^2 - 8n - 12}}$$

The general soln is,

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = C_1 2^n - 2n^2 - 8n - 12$$

Given $a_1 = 4 \quad n \geq 1$

put $n=1$ in the above,

$$a_1 = C_1 2 - 2 - 8 - 12$$

$$4 = 2C_1 - 22$$

$$2C_1 = 26 \quad \therefore C_1 = 13$$

$$\therefore \underline{\underline{a_n = 13 \cdot 2^n - 2n^2 - 8n - 12}}$$

Que:-

(Q) Solve $a_n = 4a_{n-1} + 3n2^n$, $a_0 = 4$

$$a_n - 4a_{n-1} = 3n2^n$$

homo. eqn is, $a_n - 4a_{n-1} = 0$.

$$r^n - 4r^{n-1} = 0 \Rightarrow r^{n-1}(r-4) = 0$$

$$\Rightarrow r = 4$$

Soln is, $\underline{\underline{a_n^{(h)} = C_1 4^n}}$

Corre. non homogeneous equation is,

$$a_n - 4a_{n-1} = 3n2^n \quad f(n) = n2^n$$

So we take $a_n^{(P)} = 2^n (A + Bn)$ be the soln.

$$\therefore 2^n (A + Bn) - 4 \cdot 2^{n-1} (A + B(n-1)) = 3n2^n$$

$$2^n \cdot A + 2^n \cdot n \cdot B - 2 \cdot 2^n A - 2 \cdot 2^n \cdot nB + 2 \cdot 2^n B = 3n2^n$$

$$A + nB - 2A - 2nB + 2B = 3n$$

Equating Co. of n

$$B - 2B = 3$$

$$-B = 3 \quad \therefore B = \underline{\underline{-3}}$$

Equating const term

$$-A + 2B = 0$$

$$-A - 6 = 0 \Rightarrow A = \underline{\underline{-6}}$$

$$\underline{\underline{a_n^{(P)} = 2^n (-6 - 3n)}}$$

General soln is,

$$a_n = C_1 4^n + 2^n (-6 - 3n)$$

given $a_0 = 4$

$$n=0 \Rightarrow a_0 = C_1 + 2^0(-6)$$

$$4 = C_1 - 6$$

$$C_1 = 10$$

$$\therefore a_n = \underline{\underline{10 \cdot 4^n + 2^n(-6-3n)}}$$

University questions:-

1) Solve $a_{n+2} - 4a_{n+1} + 4a_n = 2^n$, $a_0 = 1, a_1 = 1$

2) Solve $a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n$

3) Solve $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 - 2r + 1$

① Solve $a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n$

$$a_n - 4a_{n-1} + 4a_{n-2} = (n+1)2^n$$

homogeneous eqⁿ in $a_n - 4a_{n-1} + 4a_{n-2} = 0$

Let $a_n = r^n$

$$r^n - 4r^{n-1} + 4r^{n-2} = 0$$

$$r^{n-2}(r^2 - 4r + 4) = 0$$

$$(r-2)^2 = 0 \Rightarrow r = 2, 2 \text{ equal roots.}$$

C.F. in $a_n = (C_1 + C_2 n) 2^n$

$$a_n = 4 \cdot 2^n + C_2 n \cdot 2^n //$$

Solve $a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n$ — (1)

$$a_n - 4a_{n-1} + 4a_{n-2} = (n+1)2^n$$

Homogeneous $a_n - 4a_{n-1} + 4a_{n-2} = 0$ — (2)

Let $a_n = r^n$ be the soln of (2)

$$r^n - 4r^{n-1} + 4r^{n-2} = 0$$

$$r^{n-2} [r^2 - 4r + 4] = 0 \Rightarrow r^2 - 4r + 4 = 0$$

$$\Rightarrow (r-2)^2 = 0$$

$$\Rightarrow r = 2, 2$$

$$a_n^{(h)} = (C_1 + C_2 n) 2^n$$

$$a_n^{(h)} = C_1 \cdot 2^n + C_2 \cdot n \cdot 2^n \text{ be the soln.}$$

Now $f(n) = (n+1)2^n$

but 2^n and $n \cdot 2^n$ already solns of homogeneous. So we have to take $a_n^{(p)} = n^2 \cdot 2^n [A + Bn]$ as the soln of (1)

$$n^2 \cdot 2^n (A + nB) - 4 \left[(n-1)^2 \cdot 2^{n-1} (A + (n-1)B) \right] + 4 \left[(n-2)^2 \cdot 2^{n-2} (A + B(n-2)) \right] = (n+1)2^n$$

$$n^2 \cdot 2^n A + n^3 \cdot 2^n B - 4 \left[(n^2 - 2n + 1) 2^{n-1} (A + Bn - B) \right] + 4 \left[(n^2 - 4n + 4) 2^{n-2} (A + nB - 2B) \right] = (n+1)2^n$$

$$2^n \left[An^2 + Bn^3 - 2(n^2A + n^3B - n^2B - 2nA - 2n^2B + 2nB + A + nB - B) \right. \\ \left. + [n^2A + n^3B - 2n^2B - 4nA - 4n^2B + 8nB + 4A + 4nB - 8B] \right] = (n+1)2^n$$

$$An^2 + Bn^3 - 2An^2 - 2Bn^3 + 2n^2B + 4nA + 4n^2B - 4nB - 2A - 2nB + 2B \\ + An^2 + Bn^3 - 2n^2B - 4nA - 4n^2B + 8nB + 4A + 4nB - 8B = n+1$$

$$-2A - 2nB + 2B + 8nB + 4A - 8B = n + 1$$

equating co. of n both sides,

$$-2B + 8B = 1$$

$$6B = 1$$

$$\therefore B = \underline{\underline{\frac{1}{6}}}$$

equating const term, $-2A + 2B + 4A - 8B = 1$

$$2A - 6B = 1$$

$$2A - 1 = 1$$

$$2A = 2$$

$$\therefore A = \underline{\underline{1}}$$

$$a_n^{(p)} = n^2 \cdot 2^n \left(1 + \frac{1}{6}n \right)$$

$$\therefore \text{general soln is, } \underline{\underline{a_n = (C_1 + C_2 n) 2^n + n^2 \cdot 2^n \left(1 + \frac{1}{6}n \right)}}$$

(B) solve $a_n = 4a_{n-1} - 3a_{n-2} + \underline{2^n + n + 3}$ (1) $a_0 = 1, a_1 = 4$

Ans:-

we have $a_n - 4a_{n-1} + 3a_{n-2} = 2^n + n + 3$

Homo. eqⁿ is, $a_n - 4a_{n-1} + 3a_{n-2} = 0$. (2)

Let $a_n = r^n$ be the soln (2). Then

$$r^n - 4r^{n-1} + 3r^{n-2} = 0$$

$$r^{n-2} [r^2 - 4r + 3] = 0$$

$$\Rightarrow r^2 - 4r + 3 = 0$$

$$\Rightarrow (r-1)(r-3) = 0$$

$r = 1, 3$ real & distinct

$$\therefore \text{ soln is, } a_n^{(h)} = C_1 1^n + C_2 \cdot 3^n$$

$$a_n^{(h)} = \underline{C_1 + C_2 \cdot 3^n}$$

Now $f(n) = 2^n + n + 3 \therefore$ we take $a_n^{(p)}$ as

$$a_n^{(p)} = A \cdot 2^n + (Bn + C) \text{ be the soln of (1)}$$

$$\therefore (1) \Rightarrow A \cdot 2^n + (Bn + C) - 4[A \cdot 2^{n-1} + B(n-1) + C] + 3[A \cdot 2^{n-2} + B(n-2) + C] = 2^n + n + 3$$

$$A \cdot 2^n + Bn + C - 4A \cdot 2^{n-1} - 4Bn + 4B - 4C + 3A \cdot 2^{n-2} + 3Bn - 6B + 3C = 2^n + n + 3$$

$$[A \cdot 2^n - 4A \cdot 2^{n-1} + 3A \cdot 2^{n-2}] + [Bn - 4Bn + 3Bn] + C + 4B - 4C - 6B + 3C = 2^n + n + 3$$

$$2^{n-2} [4A - 8A + 3A] + n[0] + -2B = 4 \cdot 2^{n-2} + n + 3$$

equating co. of 2^{n-2} on both sides,

$$7A - 8A = 4$$

$$-A = 4 \therefore A = \underline{\underline{-4}}$$

const term on both sides,

$$-2B = 3$$

$$\therefore B = \underline{\underline{-\frac{3}{2}}}$$

$$\therefore a_n^{(p)} = -4 \cdot 2^n + \left(-\frac{3}{2}n + C\right)$$

$$\text{general soln is, } a_n = [C_1 + C_2 \cdot 3^n] + -4 \cdot 2^n + \left(-\frac{3}{2}n + C\right)$$

Generating Functions: -

The generating function of a sequence $a_0, a_1, a_2, \dots$ is the expression

$$h(x) = a_0 + a_1x + a_2x^2 + \dots = \sum_{n=0}^{\infty} a_n x^n$$

Exa: Generating fn for the sequence $1, 1, \dots$

$$h(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Generating fn for the sequence $1, 2, 3, 4, \dots$

$$h(x) = \sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + \dots$$

Generating fn for the sequence $1, a, a^2, \dots$

$$h(x) = 1 + ax + a^2x^2 + \dots = \frac{1}{1-ax}$$

Method: -

To solve a recurrence relation with initial conditions multiply the relation by an appropriate power of x and sum up suitable so as to get an explicit formula for the associated generating function. The soln of the recurrence relation is then obtained as the coefficient of x^n in the expansion of the generating function.

Q) Use the method of generating function to solve the recurrence relation
 $a_n = 3a_{n-1} + 1 \quad n \geq 1$ given $a_0 = 1$.

Ans:- Let the generating function of the sequence a_n be

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

Given the recurrence relation is,

$$a_n = 3a_{n-1} + 1 \quad \text{————— (1)}$$

Multiplying by x^n ,

$$a_n x^n = 3a_{n-1} x^n + x^n$$

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 3a_{n-1} x^n + \sum_{n=1}^{\infty} x^n$$

$$G(x) - a_0 = 3x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + \sum_{n=1}^{\infty} x^n$$

Now $a_0 = 1$

$$\therefore G(x) - 1 = 3x G(x) + x + x^2 + x^3 + \dots$$

$$G(x) - 1 = 3x G(x) + x [1 + x + x^2 + x^3 + \dots]$$

$$= 3x G(x) + x \cdot \frac{1}{1-x} \quad (\text{Using Binomial})$$

$$G(x) - 3xG(x) = \frac{1+x}{1-x}$$

$$(1-3x)G(x) = \frac{1-x+x}{1-x}$$

$$(1-3x)G(x) = \frac{1}{1-x}$$

$$\therefore G(x) = \frac{1}{(1-x)(1-3x)}$$

$$\text{Let } G(x) = \frac{A}{1-x} + \frac{B}{1-3x}$$

$$\frac{1}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x}$$

$$1 = A(1-3x) + B(1-x)$$

$$x=1 \Rightarrow 1 = -2A \quad \therefore A = -\frac{1}{2}$$

$$x=\frac{1}{3} \quad 1 = B(1-\frac{1}{3})$$

$$1 = \frac{2}{3}B \quad \therefore B = \frac{3}{2}$$

$$G(x) = \frac{-1}{2(1-x)} + \frac{3}{2(1-3x)}$$

$$= \frac{-1}{2} (1-x)^{-1} + \frac{3}{2} (1-3x)^{-1}$$

$$= \frac{-1}{2} [1+x+x^2+\dots+x^n+\dots] + \frac{3}{2} [1+(3x)+(3x)^2+\dots+(3x)^n+\dots]$$

a_n is the coefficient of x^n in $h(x)$

$$a_n = -\frac{1}{2} + \frac{3}{2}(3^n)$$

$$= \frac{3^{n+1} - 1}{2}$$

(Q. 2) Solve $a_n = 4a_{n-1} - 4a_{n-2} + 4^n$, $n \geq 2$
by using the method of generating function

given that $a_0 = 2$ $a_1 = 8$

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for the sequence a_n .

Given, $a_n = 4a_{n-1} - 4a_{n-2} + 4^n$

$$\sum_{n=2}^{\infty} a_n x^n = 4a_{n-1} x^n - 4a_{n-2} x^n + 4^n x^n$$

$$\sum_{n=2}^{\infty} a_n x^n = 4x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} - 4x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} + \sum_{n=2}^{\infty} 4^n x^n$$

$$h(x) - a_0 - a_1 x = 4x[h(x) - a_0] - 4x^2[h(x)]$$

$$+ \sum_{n=2}^{\infty} [(4x)^n - (4x)^3 + (4x)^4 - \dots]$$

$$h(x) - 2 - 8x = 4x[h(x) - 2] - 4x^2[h(x)]$$

$$+ (4x) [1 + 4x + (4x)^2 + \dots]$$

$$G(x) = 4x G(x) + 4x^2 G(x)$$

$$= 2 + 8x - 8x + (4x)^2 \cdot \frac{1}{1-4x}$$

$$G(x)[1-4x+4x^2] = 2 + \frac{16x^2}{1-4x}$$

$$\therefore G(x) = \frac{2(1-4x) + 16x^2}{(1-4x)(1-2x)^2}$$

$$G(x) = \frac{2 - 8x + 16x^2}{(1-4x)(1-2x)^2}$$

using Method of partial fraction,

$$\frac{2-8x+16x^2}{(1-4x)(1-2x)^2} = \frac{A}{1-4x} + \frac{B}{1-2x} + \frac{C}{(1-2x)^2}$$

$$2-8x+16x^2 = A(1-2x)^2 + B(1-4x)(1-2x) + C(1-4x)$$

put $x = \frac{1}{2}$,

$$2-4+4 = -C$$

$$\therefore C = \underline{\underline{-2}}$$

$$x=0 \Rightarrow 2 = A + B + C$$

$$x = \frac{1}{4} \Rightarrow 2-2+1 = \frac{1}{4}A \quad \therefore A = \underline{\underline{\frac{4}{1}}}$$

$$\therefore B = \underline{\underline{0}}$$

$$\therefore G(x) = \frac{4}{1-4x} - \frac{2}{(1-2x)^2}$$

$$G(x) = 4(1-4x)^{-1} - 2(1-2x)^{-2}$$

$$= 4[1 + (4x) + (4x)^2 + (4x)^3 + \dots] - 2[1 + 2(2x) + 3(2x)^2 + \dots + (n+1)(2x)^n]$$

$\therefore a_n = \text{Coefficient of } x^n \text{ on this generating f.}$

$$a_n = 4 \cdot 4^n - 2(n+1) \cdot 2^n$$

$$= 4^{n+1} - (n+1)2^{n+1}$$

(Q.3) Solve $u_n - u_{n-1} - 2u_{n-2} = 0$ with $u_0 = 5$ & $u_1 = -3$

$$G(x) = u_0 + u_1 x + u_2 x^2 + u_3 x^3 + \dots$$

Let $G(x) = \sum_{n=0}^{\infty} u_n x^n$ be the soln

$$\sum_{n=2}^{\infty} u_n x^n - \sum_{n=2}^{\infty} u_{n-1} x^n - \sum_{n=2}^{\infty} 2u_{n-2} x^n = 0$$

$$\therefore [u_2 x^2 + u_3 x^3 + u_4 x^4 + \dots] - [u_1 x^2 + u_2 x^3 + u_3 x^4 + \dots] - 2[u_0 x^2 + u_1 x^3 + u_2 x^4 + \dots] = 0$$

$$[u_0 - u_1(x)] - x[u_1x + u_2x^2 + u_3x^3 + \dots] - 2x^2[u_0 + u_1x^2 + u_2x^3 + \dots] = 0 \quad 259$$

$$[G(x) - u_0 - u_1(x)] - x[G(x) - u_0] - 2x^2 G(x) = 0$$

$$\text{i.e., } G(x)[1 - x - 2x^2] = u_0 + u_1x \quad \therefore u_0x$$

$$G(x) = \frac{u_0 + u_1x - u_0x}{1 - x - 2x^2}$$

$$\text{but } u_0 = 5 \\ u_1 = -3.$$

$$G(x) = \frac{5 - 8x}{1 - x - 2x^2} = -\frac{5 - 8x}{[2x^2 + x - 1]} = \frac{5 - 8x}{(2x - 1)(x + 1)}$$

$$G(x) = \frac{A}{2x - 1} + \frac{B}{x + 1}$$

$$\frac{5 - 8x}{(2x - 1)(x + 1)} = \frac{A}{2x - 1} + \frac{B}{x + 1}$$

$$8x - 5 = A(x + 1) + B(2x - 1)$$

$$x = -1 \Rightarrow -13 = -3B$$

$$\therefore B = \frac{13}{3}$$

$$x = \frac{1}{2} \Rightarrow -1 = \frac{3}{2}A$$

$$A = -\frac{2}{3}$$

$$G(x) = \frac{-\frac{2}{3}}{2x - 1} + \frac{\frac{13}{3}}{x + 1}$$

$$= \frac{-\frac{2}{3}}{-(1-2x)} + \frac{\frac{13}{3}}{1+x}$$

$$= \frac{2}{3} (1-2x)^{-1} + \frac{13}{3} (1+x)^{-1}$$

$$= \frac{2}{3} [1 + 2x + (2x)^2 + (2x)^3 + \dots + (2x)^n] + \frac{13}{3} [1 - x + x^2 - x^3 + \dots + (-1)^n x^n]$$

$$a_n = \text{coe. of } x^n$$

$$= \frac{2}{3} \cdot 2^n + \frac{13}{3} (-1)^n$$

$$= \frac{2^{n+1}}{3} + \frac{13}{3} (-1)^n$$

(9.3) Solve $a_n - 3a_{n-1} + 2 = 0$ $a_0 = 1, n \geq 1$

using generating fn.

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating fn of the sequence $\dots a_n$

given, $a_n - 3a_{n-1} + 2 = 0$.

$\Rightarrow a_n x^n - 3a_{n-1} x^n + 2x^n = 0$.

$$\sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} 3a_{n-1} x^n + \sum_{n=1}^{\infty} 2x^n = 0$$

$$\sum_{n=1}^{\infty} a_n x^n - 3x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + 2 \sum_{n=1}^{\infty} x^n = 0$$

$$[G(x) - a_0] - 3x G(x) + 2[x + x^2 + x^3 + \dots] = 0$$

given $a_0 = 1$

$$G(x) - 1 - 3x G(x) + 2x[1 + x + x^2 + x^3 + \dots] = 0$$

$$\begin{aligned} (1 - 3x)G(x) &= 1 - 2x(1 + x + x^2 + \dots) \\ &= 1 - 2x \cdot \frac{1}{1-x} \end{aligned}$$

$$(1-3x)G(x) = \frac{1-x-2x}{1-x}$$

$$(1-3x)G(x) = \frac{(1-3x)}{1-x}$$

$$\therefore G(x) = \frac{1}{1-x} = 1+x+x^2+\dots$$

$\therefore a_n = \text{the co. of } x^n \text{ in the exp. of } G(x)$

$$a_n = \frac{1}{x}$$

$$\boxed{\frac{1}{x}}$$

3) Solve $a_n = 2a_{n-1} + 2^n$ with $a_0 = 2$ using generating functions.

$$\text{Let } G(x) = \sum a_n x^n$$

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 2a_{n-1} x^n + \sum_{n=1}^{\infty} 2^n x^n$$

$$\sum_{n=1}^{\infty} a_n x^n = 2x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + \sum_{n=1}^{\infty} (2x)^n$$

$$[G(x) - a_0] = 2x G(x) + [2x + (2x)^2 + \dots]$$

$$\ln(x) - 2 = 2x \ln(x) + 2x[1 + 2x + (2x)^2 + \dots]$$

$$(1-2x)\ln(x) = 2 + 2x \cdot \frac{1}{(1-2x)}$$

$$= \frac{2-4x+2x}{1-2x}$$

$$(1-2x)\ln(x) = \frac{2-2x}{1-2x}$$

$$\therefore \ln(x) = \frac{2-2x}{(1-2x)^2}$$

using Method of partial fraction

$$\frac{2-2x}{(1-2x)^2} = \frac{A}{1-2x} + \frac{B}{(1-2x)^2}$$

$$2-2x = A(1-2x) + B$$

$$\frac{1}{2} \Rightarrow 2-1 = 0 + B$$

$$\therefore \underline{\underline{1 = B}}$$

$$\Rightarrow 2 = A + B \quad \therefore \underline{\underline{A = 1}}$$

$$\frac{2-2x}{(1-2x)^2} = \frac{1}{1-2x} + \frac{1}{(1-2x)^2}$$

$$G(x) = (1-2x)^{-1} + (1-2x)^{-2}$$

We have

$$(1-x)^{-1} = 1 + x + x^2 + \dots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$G(x) = 1 + (2x) + (2x)^2 + (2x)^3 + \dots$$

$$+ 1 + 2(2x) + 3(2x)^2 + 4 \cdot (2x)^3 + \dots$$

$a_n =$ co. of x^n in the exp. of $G(x)$

$$= 2^n + (n+1) \cdot 2^n$$

$$= 2^n [1 + n + 1]$$

$$\underline{\underline{a_n = 2^n (n+2)}}$$

GROUPS

Module V

Binary operations:- [reference:- kolman / Busby / zoss)
discrete mathematical structures]

An operation or composition on a set A is an everywhere defined function $f: A \times A \rightarrow A$. Also, a binary operation should satisfy the following properties

- (i) Since $\text{dom}(f) = A \times A$ f assigns an element $f(a, b)$ of A to each ordered pair (a, b) in $A \times A$. i.e., the binary operation must be defined for each ordered pair (a, b) of element of A .
- (ii) If a & b are elements in A then $a * b \in A$.

Usually we use the symbol $a * b$ to denote $f(a, b)$.

exa: 1 Let $A = \mathbb{Z}$ define $a * b = a + b$ then $*$ is a binary operation.

exa: 2 Let $A = \mathbb{R}$ $a * b = a/b$. then $*$ is not a binary operation for exa: $3 * 0$ is not defined.

' $*$ ' is a binary operation on the sets of non zero real no.

exa: 3 Let $A = \mathbb{Z}^+$ (positive integers) define $a * b = a - b$ then $*$ is not a binary operation.

exa: 4 Let $A = \mathbb{Z}$ define $a * b = \max\{a, b\}$ Then $*$ is a bi. op.

exa: 5 Let $A = P(S)$ for some set S . If v and w are subsets of S define $v * w$ as $V \cup W$ then $*$ is a binary op.

Properties:-

1) A binary operation $*$ on a nonempty set G is said to be associative if
 $(a * b) * c = a * (b * c)$ $\forall a, b, c$ in G .

2) A binary operation $*$ on a nonempty set G is said to be commutative if
 $a * b = b * a$ for any a, b in G .

- exa: -1 The usual operations of addition or multiplication on the sets of integers is both associative and commutative.
- exa: 2 The operations of $\cup$ or $\cap$ on the sets of subsets of a set is both associative and commutative.
- exa: 3 The binary operation of subtraction on $\mathbb{Z}$ is not commutative since $2-3 \neq 3-2$

Definition: -

Let G be a nonempty set with binary operation. An element 'e' is called identity or unit element if,

$$a * e = e * a = a \quad \text{for each } a \in G.$$

exa: On The set of integers under addition, 0 is the identity element.

exa: Consider the set of subsets of a set with ' $\cup$ ' as operation then the empty set ϕ is the identity element. If ' $\cap$ ' is the operator then the whole set X is the identity element.

Definition: -

Let G be a nonempty set with binary operation $*$ and identity element 'e' then an element $a \in G$ is said to have an inverse w.r.to $*$ if there exists another element $a' \in G$ such that

$$a * a' = e = a' * a.$$

exa: In real numbers, inverse of a w.r.to addition is -a. If we consider nonzero real numbers then the inverse of a w.r.to $\times$ is $\frac{1}{a}$.

Semi Group

A semi group is a nonempty set S together with an associative binary operation $*$ defined on S . We denote the semi group by $(S, *)$

exa: $(\mathbb{Z}, +)$ is a semi group. But $(\mathbb{Z}, -)$ is not a semi group since '-' is not-associative.

Monoid:-

A monoid is a semi group that has an identity.

exa: The semigroup $P(S)$ has the identity ϕ under the operation $(\cup)$. So $(P(S), \cup)$ is a monoid.

Groups:-

A group $(G, *)$ is a nonempty set G , with binary operation $*$ satisfying the following conditions.

- (i) $(a * b) * c = (a * (b * c)) \quad \forall a, b, c \in G$ [associativity]
- (ii) There exist an element $e \in G$ such that $a * e = a = e * a \quad \forall a \in G$. [existence of identity]
- (iii) For each $a \in G$ $\exists$ an element $a' \in G$ such that $a * a' = e = a' * a$ [existence of inverse]

exa:- (1) The set R of real numbers with addition. Similarly $(Z, +)$, $(Q, +)$ is a group.

(2) The set R^* of non zero real numbers with multiplication is a group.

exa: (3) $\{1, -1, i, -i\}$ is a group for multiplication.

Abelian Group

A group G is said to be Abelian if $a * b = b * a$ for all a, b in G .

exa: $(R, +)$, $(Z, +)$, $(Q, +)$ all are abelian groups.

Properties of Groups:- The identity element in a gp is unique.

1. Theorem:1

Let G be a group. Each element a in G has only one inverse in G .

Proof:- Let a' and a'' be inverses of a . Then, we know

$$a * a' = e = a' * a$$

$$a * a'' = e = a'' * a$$

$$\text{Consider } a'(a a'') = a' e = a'$$

$$(a' a) a'' = e a'' = a''$$

Since G is a group it holds associative property. Hence

$$\underline{a' = a''}. \quad \text{Thus in a group } G, \text{ each element has a unique inverse.}$$

Theorem :- 2

Let G be a group and let $a, b, c \in G$. Then

- (i) $ab = ac$ implies $b = c$ [left cancellation]
 (ii) $ba = ca$ implies $b = c$ [right cancellation]

Proof:- (i) Suppose $ab = ac$
 $\times$ with a^{-1} on the left.

(ii) similar proof.

$$a^{-1}(ab) = a^{-1}(ac)$$

$$(a^{-1}a)b = (a^{-1}a)c$$

$$eb = ec$$

$$\underline{b = c}$$

Theorem 3 Let G be a group and $a, b \in G$. Then

$$(a) \underline{(a^{-1})^{-1} = a} \quad (b) \underline{(ab)^{-1} = b^{-1}a^{-1}}$$

Proof:- (a) Here we show that the element a acts as an inverse for a^{-1} . Then $(a^{-1})^{-1}a^{-1} = e$ and $a^{-1}a = e$.
 Also $a \in G \Rightarrow a^{-1} \in G$. Let $(a^{-1})^{-1}$ be the inverse of a^{-1} .
 $a^{-1}a = a a^{-1} = e$ and $(a^{-1})^{-1}a^{-1} = e \Rightarrow ((a^{-1})^{-1}a^{-1})a = ea$
 $(a^{-1})^{-1}(a^{-1}a) = a$

Since the inverse of an element is unique $\underline{(a^{-1})^{-1} = a}$

(b) To prove (b) use the defn of inverse

$$\begin{aligned} \text{Consider } (ab)(b^{-1}a^{-1}) &= a(b(b^{-1}a^{-1})) \\ &= a(bb^{-1}a^{-1}) \\ &= a(ea^{-1}) \\ &= aa^{-1} \\ &= e \end{aligned}$$

similarly

$$(b^{-1}a^{-1})(ab) = e$$

$$\therefore \underline{(ab)^{-1} = b^{-1}a^{-1}}$$

Theorem 4 Let G be a group and let $a, b \in G$. Then

(a) The eqⁿ $ax=b$ has unique solution in G

(b) The eqⁿ $ya=b$ has unique solution in G .

Proof:- Consider the eqⁿ $ax=b$. Multiplying both sides by a^{-1} then,

$$a^{-1}(ax) = a^{-1}(b)$$

$$ex = a^{-1}b$$

i.e., $x = a^{-1}b$ is a solution of the eqⁿ $ax=b$.

Now we show that the solution is unique. Suppose

x_1, x_2 are two solutions then

$$ax_1 = b = ax_2 \quad \text{By left cancellation law we get } x_1 = x_2.$$

Similarly we can p.T the eqⁿ $ya=b$ has a unique solⁿ in G .

Defn:- If G is a group that has finite no. of elements

we say that G is a finite group.

Defn:- The order of a group G is the no. of elements in G and it is denoted by $O(G)$ or $|G|$.

Exa: 1 The set $\{1, -1, i, -i\}$ is a finite group of order 4.

$x \cdot$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Exa: 2 The set $\{e\}$ is a group of order 1.

Theorem 5 If every element of a group G is its own inverse then G is abelian.

Proof:- Let $a, b \in G$. Since every element of G is its own inverse we have $a^{-1} = a, b^{-1} = b$

Also as $ab \in G$ we get $(ab)^{-1} = ab$.

$$\text{i.e., } b^{-1}a^{-1} = ab \quad (\text{since } (ab)^{-1} = b^{-1}a^{-1})$$

$$\text{so, } ba = ab$$

Thus G is abelian.

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(Q) 3 prove that a group with the property that $x^2 = e$ for every element x is necessarily abelian.

proof:- let $a, b \in G$ then $a * a = e$ $b * b = e$
 $\text{i.e., } a^{-1} = a$ $b^{-1} = b$.

$$\begin{aligned} \text{Then if } ab \in G \text{ then we get } ab * ab &= a(b * ab) = e \\ &= a(bba) \\ &= a(ea) \\ &= a^2 \\ &= e // \end{aligned}$$

$$\text{Thus, } (ab)^{-1} = ab.$$

$$\text{i.e., } b^{-1}a^{-1} = ab$$

$$\underline{ba = ab.} \quad \text{Thus } G \text{ is abelian.}$$

(Q) If G is a group of even order prove that it has an element a different from the identity element such that $a^2 = e$.
 Since every element in the group possess a unique inverse and the identity 'e' is its own inverse, the hypothesis that G is of even order demands that there exists at least one element $a \neq e$ in G such that it is its own inverse. $\text{i.e., } a^{-1} = a$. Hence $a^2 = \underline{e}$.
 $a^2 = aa = aa^{-1} = \underline{e}$.

Subgroups

A nonempty subset H of a group G is called a subgroup of G if H itself forms a group w.r.to the operation already defined on G .

i.e., if (i) whenever $a, b \in H$ then $ab \in H$

(ii) The identity e of G belongs to H

(iii) For each $a \in H$ its inverse a^{-1} in G also belongs to H

Ex-1 For any group G , G is a subgroup of itself. Similarly the trivial group $\{e\}$ is a subgroup of G .

Any subgroup which is not equal to G is called a proper subgroup of G .

Ex-2 The group of integers under addition is subgroup of rational nos under addition.

$(\mathbb{Q}, +)$ is a subgroup of $(\mathbb{R}, +)$

Ex-3 The additive gp of even integers is a subgroup of the additive gp of integers.

$\{1, -1\}$ is a subgroup of $\{1, -1, i, -i\}$.

Theorem:-

A nonempty subset H of a group G is a subgroup iff whenever $a \in H$ $b \in H$ the product ab^{-1} also belongs to H .

Proof:- Assume that H is a subgroup of G . Then if $a \in H$ and $b \in H$ by the axiom (iii) of groups $b^{-1} \in H$.

So by closure property $ab^{-1} \in H$.

Conversely assume that $ab^{-1} \in H$. choose some $a \in H$

Then by taking $b=a$ in the given condition

we have $a a^{-1} \in H$ i.e. $e \in H$.

Since $e \in H$ for any $a \in H$ $e * a^{-1} = a^{-1} \in H$ This verifies condition (iii)

Thus for any $a \in H$ $b \in H$, we get $b^{-1} \in H$

Hence $a \in H, b^{-1} \in H \Rightarrow a(b^{-1})^{-1} \in H$

$\Rightarrow ab \in H$. This verifies closure property.

Hence H is a subgroup of G .

Permutations

permutation of a set is a rearrangement of the elements of that set.

for exa: $\{1, 2, 3, 4, 5\}$ a rearrangement of the elements of this set can be write as $\{4, 2, 5, 3, 1\}$ i.e., schematically we can write

$$\begin{array}{l} 1 \rightarrow 4 \\ 2 \rightarrow 2 \\ 3 \rightarrow 5 \\ 4 \rightarrow 3 \\ 5 \rightarrow 1 \end{array}$$

So we can say that permutation of a set is a mapping from the set to itself in which, each element has distinct images in the set.

Defn:- A permutation of a set A is a function from A into A such that it is both one-one and onto.

Note:- If A is a finite set $\{1, 2, \dots, n\}$ then the group of all permutations of A is symmetric group on n letters and is denoted by S_n . S_n has $n!$ elements.

exa:- Symmetric group S_3 . or the group S_3 of symmetries of an equilateral triangle.

The symbol S_i used for rotations and μ_0 for mirror images in bisectors of angles

$$S_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$S_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$



$\therefore S_3 = \{S_0, S_1, S_2, \mu_0, \mu_1, \mu_2\}$ forms a group.

	S_0	S_1	S_2	μ_1	μ_2	μ_3
S_0	S_0	S_1	S_2	μ_1	μ_2	μ_3
S_1	S_1	S_2	S_0	μ_2	μ_3	μ_1
S_2	S_2	S_0	S_1	μ_3	μ_1	μ_2
μ_1	μ_1	μ_3	μ_2	S_0	S_2	S_1
μ_2	μ_2	μ_1	μ_3	S_1	S_0	S_2
μ_3	μ_3	μ_2	μ_1	S_2	S_1	S_0

$$S_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

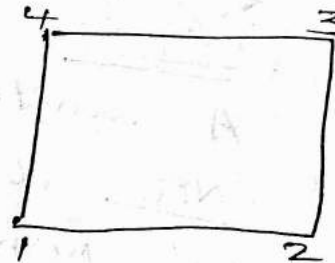
$$\mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

clearly it is a group having the identity S_0 and it is non abelian.

na: Dihedral group D_4 of permutations corresponding to the ways that two copies of a square with vertices 1 2 3 4 can be placed one covering the other.

Here we are using S_i for rotations, μ_i for mirror images in perpendicular bisectors of sides and δ_i for diagonal flips.



$$S_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad S_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \quad S_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\delta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \quad \delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

$\therefore \{S_0, S_1, S_2, S_3, \mu_1, \mu_2, \delta_1, \delta_2\}$ forms a group.

D_4 is also non abelian.

Cyclic groups.

A group is said to be cyclic if it is capable of being generated by a single element. The single element is called the generator of the group.

The gp $H = \{a^n \mid n \in \mathbb{Z}\}$ is called the cyclic gp generated by a . A cyclic group can have more than one generators.

exa ① The group $\mathbb{Z}$ of integers is cyclic $\mathbb{Z} = \langle 1 \rangle = \langle -1 \rangle$
 1 and -1 are generators of $\mathbb{Z}$.

exa ② The group $G = \{1, -1, i, -i\}$ is cyclic with i and $-i$ as generators.

③ The group $\mathbb{Z}_n$ is cyclic with generator 1 .

Theorem:-

Let G be a group and $a \in G$. Then the set $H = \{a^n \mid n \in \mathbb{Z}\}$ is a subgroup of G ~~and is the smallest subgroup of G containing a .~~

Proof:- $H = \{a^n \mid n \in \mathbb{Z}\}$

Let $x \in H$ and $y \in H$. Then $x = a^m$ & $y = a^n$ for some $m, n \in \mathbb{Z}$

$$\begin{aligned} \text{Then } xy^{-1} &= a^m \cdot (a^n)^{-1} \\ &= a^m (a^{-1})^n \\ &= a^{m-n} \in H. \end{aligned}$$

$\therefore x \in H, y \in H \Rightarrow xy^{-1} \in H$. Hence H is a subgroup of G .

Theorem:-

If G is a cyclic gp, then G is abelian.

Ans:- Let $G = \langle a \rangle$ be cyclic.

If $x, y \in G$ then $x = a^m$ $y = a^n$. Then

$$\begin{aligned} xy &= a^m \cdot a^n \\ &= a^{m+n} \\ &= a^{n+m} \\ &= yx. \end{aligned}$$

$\therefore xy = yx$
Hence G is abelian.

Coset

Let $(G, *)$ be a subgroup of $(G, *)$. Then

$H * a = \{ h * a \mid h \in H, a \in G \}$ is called the right coset of H in G . $H * a$ is usually denoted as Ha .

Similarly $a * H = \{ a * h \mid h \in H, a \in G \}$ is the left coset of H in G it is also denoted as aH .

Exa:- Let $G = \{ 1, -1, i, -i \}$ ^{be a grp under xcation} $H = \{ 1, -1 \}$ be a sub grp of G . Then

$$H(1) = \{ 1 \cdot 1, 1 \cdot (-1) \} = \{ 1, -1 \}$$

$$H(-1) = \{ 1 \cdot (-1), (-1) \cdot (-1) \} = \{ -1, 1 \}$$

$$H(i) = \{ 1 \cdot i, (-1) \cdot i \} = \{ i, -i \}$$

$$H(-i) = \{ 1 \cdot (-i), (-1) \cdot (-i) \} = \{ -i, i \}$$

Notes:-

① Here clearly $H1$ and $H(-1)$ are two distinct right coset of H in G .

② $H \cup H(i) = G$. i.e., G is partitioned into two distinct right coset of H in G

③ $H \cap H(i) = \emptyset$

Any two right coset of H in G are either identical or disjoint

- (4) The right coset of H in G are also subsets of H in G
- (5) H itself is a right coset and the no. of elements in each right coset is the same as the no. of elements in H .
- (6) If $a \in H$, then $Ha = H$
If $a \notin H$ & $a \in G$ then $Ha \neq H$
- (7) In a similar way we can form the left coset of H in G and the above two results are true.

Theorem

ex: $Z_6 = \{0, 1, 2, 3, 4, 5\}$ under $+6$.

$H = \{0, 2, 4\}$ is a subgroup. Find the right & left coset of H in G .

$$H_0 = \{0, 2, 4\}$$

$$H_3 = \{3, 5, 1\}$$

$$H_5 = \{5, 1, 3\}$$

$$H_1 = \{1, 3, 5\}$$

$$H_4 = \{4, 0, 2\}$$

$$H_2 = \{2, 4, 0\}$$

Theorem Let $(G, *)$ be a grp and $(H, *)$ be a subgroup. Any two right cosets of H in G are either identical or disjoint.

Proof:- Let $a, b \in G$ and Ha, Hb be two right cosets of H in G .

Let $Ha \cap Hb \neq \emptyset$.

$\therefore$ There exists an element $c \in Ha \cap Hb$.

$\Rightarrow c \in Ha$ and $c \in Hb$

$\therefore$ There exists $h_1, h_2 \in H$ such that

$$c = h_1 * a \quad , \quad c = h_2 * b$$

$\therefore$ we can write

$$h_1 * a = h_2 * b$$

pre Multiply with h_2^{-1} and post multiply with a^{-1}

$$h_2^{-1} * h_1 * a * a^{-1} = h_2^{-1} * h_2 * b * a^{-1}$$

$$h_2^{-1} * h_1 = b * a^{-1}$$

but $h_2^{-1} * h_1 \in H \quad \therefore b * a^{-1} \in H$.

$$\text{Hence } H = H(b * a^{-1})$$

if $a \in H$
★ $Ha = H$

$$H * a = H(b * a^{-1} * a)$$

$$H * a = H * b$$

$$\Rightarrow \underline{Ha = Hb}$$

Lagrange's theorem

Let H be a subgroup of a finite group G .
Then the order of H is a divisor of the order of G . i.e., $O(H) \mid O(G)$.

Proof:- Let G be a grp of finite order n .
 $O(G) = n$. Let H be a subgroup of G and

$$O(H) = m.$$

Suppose $h_1, h_2, \dots, h_m$ are ' m ' elements of H .

Let $a \in G$. Then Ha is the right coset of H in G and we have

$$Ha = \{h_1 a, h_2 a, \dots, h_m a\}$$

Ha has ' m ' distinct elements, because

$$\text{if } h_i a = h_j a$$

$$\Rightarrow h_i = h_j$$

$\therefore$ Each right coset of H in G has m distinct elements. Any two distinct right cosets of H in G are disjoint.

$\therefore$ They have no element in common.
 Since G is finite, ~~no~~ ^{number} of distinct right coset of H in G will be finite say k .
 The union of these k distinct right cosets of H in G is equal to G .

$$\text{i.e., } G = Ha_1 \cup Ha_2 \cup \dots \cup Ha_k$$

$$\begin{aligned} \text{no. of elements in } G &= \text{no. of elements in } Ha_1 + \\ &\text{no. of element in } Ha_2 + \dots + \text{no. of elements in } Ha_k \end{aligned}$$

$$\Rightarrow o(G) = m + m + \dots + m \text{ } k \text{ times}$$

$$o(G) = km$$

$$n = km$$

$$k = \frac{n}{m}$$

$$\Rightarrow m \text{ is a divisor of } n$$

$$\Rightarrow o(H) \text{ is a divisor of } o(G)$$

Homomorphisms and Isomorphisms of Groups

Homomorphism: - Suppose G and G' are two groups, the composition in each be $*$ and $\cdot$ respectively then

A mapping $f: (G, *) \rightarrow (G', \cdot)$ is said to be a homomorphic mapping of G into G' if

$$f(a * b) = f(a) \cdot f(b) \quad \forall a, b \in G.$$

Isomorphism

Let $(G, *)$ and $(G', \cdot)$ are two groups.

The a mapping $f: G \rightarrow G'$ is called an isomorphic mapping of G into G'

if (i) f is 1-1 i.e., distinct elements in G have distinct f images in G' .

$$(ii) \quad f(a * b) = f(a) \cdot f(b) \quad \forall a, b \in G$$

Thus isomorphism is a special type of homomorphism. In other words we

can say that a one-one homomorphic mappings are called isomorphisms.

Theorem:- (1) Let f be a homomorphic mapping of a group G into G' . Then

$$(i) f(e) = e' \quad \text{where } e \text{ is the identity in } G \\ e' \quad \text{''} \quad \text{in } G'$$

$$(2) f(a^{-1}) = [f(a)]^{-1} \quad \text{where } a^{-1} \text{ is the inverse of } a \text{ in } G.$$

$[f(a)]^{-1}$ is the " of $f(a)$ in G'

Proof:- Let e be the identity of G and e' be the identity of G' . Let $a \in G$ then

$$f(a) \in G'$$

$$\text{Now, } e' f(a) = f(a) \quad \left[\begin{array}{l} \text{since } e' \text{ is identity} \\ \text{in } G' \end{array} \right]$$

$$= f(\overset{e+a}{\cancel{e+a}}) \quad e \text{ is the identity of } G$$

$$e' f(a) = f(e) f(a) \quad \left[\text{since } f \text{ is isomorphism} \right]$$

$$\Rightarrow e' = f(e) \quad \text{by right cancellation law.}$$

$\therefore f(e)$ is identity of G'

(2) Let e is the identity of G and
 e' " " of G'

Then $f(e) = e'$

Let $a \in G$ then $a^{-1} \in G$ and

$$a a^{-1} = e$$

we have $e' = f(e)$

$$e' = f(a a^{-1})$$

$$e' = f(a) f(a^{-1})$$

$\therefore f(a^{-1})$ is the inverse of $f(a)$

$$\therefore \underline{\underline{f(a^{-1}) = [f(a)]^{-1}}}$$

Properties of cyclic group

1. A cyclic group is abelian.

Proof: Let $\{G, *\}$ be a cyclic group with $a \in G$ as generator.

Let $b, c \in G$ Then $b = a^m$ & $c = a^n$ where m, n are integers.

$$\begin{aligned} \text{Now } b * c &= a^m * a^n = a^{m+n} \\ &= a^{n+m} \end{aligned}$$

m, n are integers

$$\begin{aligned} &= a^n * a^m \\ &= c * b. \end{aligned}$$

Hence $\{G, *\}$ is abelian.

If 'a' is a generator of a cyclic group $\{G, *\}$ then a^{-1} is also generator of $\{G, *\}$.

Let $b \in G$ then $b = a^m$ where 'm' is an integer

$$b = (a^{-1})^{-m} \text{ where } -m \text{ is an integer}$$

a^{-1} is also a generator of $\{G, *\}$

Direct Product of Groups

Suppose we have any 2 groups G_1 and G_2 . These 2 groups can be finite or infinite, abelian or non abelian. The direct product is a way to combine these 2 groups into a new, larger group. It's the set of all pairs where the first component is from G_1 and second from G_2 .

$$G_1 \times G_2 = \{(x, y) \mid x \in G_1, y \in G_2\}$$

Notation is same as Cartesian product of 2 sets

Exa:- Let $G_1 =$ set of all integers under addition
($\mathbb{Z}, +$)

$$G_2 = \{1, -1, i, -i\} \text{ under } \times$$

$$G_1 \times G_2 = \{(x, y) \mid x \in \mathbb{Z}, y = \pm 1 \text{ or } \pm i\}$$

Note:-

group operation must be done on component wise.

• Let $(a, b), (x, y) \in G_1 \times G_2$

• $(a, b) \cdot (x, y) = (a \cdot x, b \cdot y)$

• Identity = (e_1, e_2) $\left. \begin{matrix} a \in G_1 & e_2 \in G_2 \end{matrix} \right\}$

$$(1, -1) \cdot (-3, i) = (1 \cdot -3, -1 \cdot i)$$

$$= (-4, -i)$$

$$e = (0, 1) \quad \text{since } (5, -i) \cdot (0, 1) = (5 + 0, -i \cdot 1) = (5, -i)$$

What is the inverse of $(12, -i)$

$$(12, -i)(a, b) = (0, 1)$$

$$12 + a = 0$$

$$a = \underline{\underline{-12}}$$

$$-i \cdot b = 1$$

$$b = \frac{1}{-i} \times \frac{i}{i} = \underline{\underline{+i}}$$

∴ Inverse is, $(-12, i)$

Direct product of more ^{than} 2 Groups

$$G_1 \times G_2 \times G_3 = \{(x, y, z) \mid x \in G_1, y \in G_2, z \in G_3\}$$

$$(a, b, c) \cdot (x, y, z) = (a \cdot x, b \cdot y, c \cdot z)$$

$$\text{Identity} = (e_1, e_2, e_3)$$

(8) If $S = \mathbb{N} \times \mathbb{N}$, the set of ordered pairs of positive integers with the operation $*$ defined by

$$(a, b) * (c, d) = (ad + bc, bd) \text{ and}$$

if $f: (S, *) \rightarrow (\mathbb{Q}, +)$ defined by $f(a, b) = \frac{a}{b}$

S.T f is a semigroup homomorphism.

Ans: First we have to S.T $(S, *)$ is a semigroup.

Let $(a, b), (c, d), (e, f) \in S$

$$\begin{aligned} \text{Then } [(a, b) * (c, d)] * (e, f) &= (\underbrace{ad + bc}_{\text{from } (a, b) * (c, d)}, \underbrace{bd}_{\text{from } (a, b) * (c, d)}) * (e, f) \\ &= ((ad + bc)f + bde, bdf) \\ &= (adf + bcf + bde, bdf) \quad \text{--- (1)} \end{aligned}$$

$$\begin{aligned} (a, b) * [(c, d) * (e, f)] &= (a, b) * (\underbrace{cf + de}_{\text{from } (c, d) * (e, f)}, \underbrace{df}_{\text{from } (c, d) * (e, f)}) \\ &= (adf + b(cf + de), bdf) \\ &= (adf + bcf + bde, bdf) \end{aligned}$$

$\therefore$ From (1) & (2) $*$ is associative. (2)

$f: (S, *) \rightarrow (Q, +)$ defined by

$$f(a|b) = \frac{a}{b} \quad \text{we have to s.t.}$$

f is a homomorphism.

i.e., let $(a, b), (c, d) \in S$.

$$f[(a, b) * (c, d)] = f[ad+bc, bd]$$

$$= \frac{ad+bc}{bd}$$

$$= \frac{ad}{bd} + \frac{bc}{bd}$$

$$= \frac{a}{b} + \frac{c}{d}$$

$$f(a * b) = f(a) \cdot f(b)$$

$$a, b \in X$$

$$\text{since } f(a|b) = \frac{a}{b}$$

$$f[(a, b) * (c, d)] = f(a|b) + f(c|d)$$

$\therefore f$ is a homomorphism.

Hence $f: (S, *) \rightarrow (Q, +)$ is a semi-group.

homomorphism

(9) If $S = \{0, 1, 2, 3\}$ is a subset of the semigroup $\{\mathbb{Z}_4, +_4\}$ and $T = \{1, 3, 7, 9\}$ is a subset of the semigroup $\{\mathbb{Z}_{10}, \times_{10}\}$. If f is a function $f: S \rightarrow T$ defined by $f(0) = 1$, $f(1) = 3$, $f(2) = 9$ & $f(3) = 7$. S-T f is an isomorphism.

Ans: Composition table, is

S:

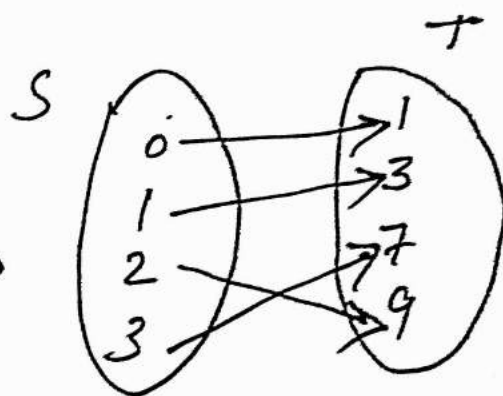
$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\times_{10}$	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

$$f: S \rightarrow T$$

clearly f is 1-1 and onto.

Since every distinct elements of S has distinct images



and every element of T possess a pre image, i.e.

Now we have to S-T f is a homomorphism

So we have to S-T

$$f(a +_4 b) = f(a) \times_{10} f(b)$$

$$f\left(1 + \frac{2}{4}\right) = f(3) = \underline{\underline{7}}$$

$$f(1) \times_{10} f(2) = 3 \times_{10} 9 = 27 \equiv \underline{\underline{7}}$$

Similarly

$$f(1 + 3) = f(0) = \underline{\underline{1}}$$

$$f(1) \times_{10} f(3) = 3 \times_{10} 7 = \underline{\underline{1}}$$

and so on further combinations

$$\therefore f\left(a + \frac{b}{4}\right) = f(a) \times_{10} f(b) \quad \forall a, b \in$$

$\therefore f$ is an isomorphic mapping

University Questions

- (i) S.T. the set N of natural numbers is a semigroup under the operation $x * y = \max(x, y)$.
Is it a monoid?

Ans:

Let $x, y \in N$

$$x * y = \max(x, y) \in N$$

closure property satisfied.

Let $a, b, c \in N$

$$\begin{aligned} a * (b * c) &= a * \max\{b, c\} \\ &= \max\{a, \max\{b, c\}\} \\ &= \max\{a, b, c\} \\ &= \underline{\underline{\max\{a, b, c\}}} \end{aligned}$$

$$\begin{aligned} (a * b) * c &= \max\{a, b\} * c \\ &= \max\{\max\{a, b\}, c\} \\ &= \underline{\underline{\max\{a, b, c\}}} \end{aligned}$$

$$\therefore a * (b * c) = (a * b) * c$$

$\therefore *$ is associative. Hence $(N, *)$ is a Semigroup.

Hence 1 is the identity.

Let $a \in \mathbb{N}$ where $a \geq 1$

$$\text{then } a * 1 = \max\{a, 1\}$$

$$= \underline{\underline{a}}$$

$\therefore$ identity exists

$\therefore \{ \mathbb{N}, * \}$ is a monoid.

$$a * b = \underline{\underline{\max\{a, b\}}} = \underline{\underline{1}}$$

b is called the inverse of a iff

$$\boxed{a * b = e}$$

(U.Q) Let $G = \{1, a, a^2, a^3\}$ ($\boxed{a^4 = 1}$) and
 $H = \{1, a^2\}$ is a subgroup of G under
 multiplication. Find all cosets of H .

$a * H$, $H * a$
 (left) (right)

Let $1 \in G$

defn: $\boxed{H * a = \{h * a \mid h \in H, a \in G\}}$

$$H * 1 = \{1, a^2\} = H$$

if $a \in G$ and $a \in H$ also.
 $\boxed{H * a = H}$

Second element

Let $a \in G$.

$$G = \{1, a, a^2, a^3\}$$

$$H = \{1, a^2\}$$

$$\therefore H * a = \{a \cdot 1, a \cdot a^2\}$$

$$H * a = \underline{\underline{\{a, a^3\}}}$$

Next

~~Let~~ $a^2 \in G$

$$H * a^2 = \{a^2 \cdot 1, a^2 \cdot a^2\}$$

$$= \{a^2, 1\} \quad \text{Since } a^4 = 1$$

$$\underline{\underline{= H}}$$

$a^3 \in G$, then

$$H * a^3 = \{a^3 \cdot 1, a^3 \cdot a^2\}$$

$$= \{a^3, a^4 \cdot a\}$$

$$\underline{\underline{= \{a^3, a\}}}$$

distinct

$$\therefore \underline{\underline{\text{Cosets are } \{a, a^3\}, \{1, a^2\}}}$$

- (6) P.T the intersection of 2 subgroups of a group G is also a subgroup of G .
 Give an example to s.t the union of 2 subgroups of G need not be a subgroup of G .

Let G be a group.

$H \subseteq G$ and H is called subgroup of G ,

$$\text{iff } \boxed{\forall a, b \in H \Rightarrow a * b^{-1} \in H.}$$

P. proof:-

Let H_1 and H_2 be any 2 subgroups of G .

Then $H_1 \cap H_2$ is nonempty, since identity element belongs to both H_1 & H_2

We have to s.t $H_1 \cap H_2$ is a subgroup

Let $a \in H_1 \cap H_2 \Rightarrow$ Then $a \in H_1$ & $a \in H_2$

$b \in H_1 \cap H_2 \Rightarrow b \in H_1$ & $b \in H_2$

Since H_1 is a subgroup,

if $a \in H_1$, $b \in H_1$ then $a * b^{-1} \in H_1$

Similarly

if $a \in H_2$ & $b \in H_2$ then $a * b^{-1} \in H_2$ since

H_2 is a subgroup.

$$\therefore a \times b^{-1} \in H_1 \cap H_2$$

$$a, b \in H_1 \cap H_2 \implies a \times b^{-1} \in H_1 \cap H_2$$

Hence $H_1 \cap H_2$ is a subgroup

Let G be the additive group of integers

$$G = (\mathbb{Z}, +)$$

$$H_1 = \{ \dots -6, -4, -2, 0, 2, 4, 6, \dots \}$$

$$H_2 = \{ \dots -9, -6, -3, 0, 3, 6, 9, \dots \}$$

clearly H_1 & H_2 are 2 subgroups of $(\mathbb{Z}, +)$

$$H_1 \cup H_2 = \{ \dots -9, -6, -4, -3, -2, 0, 2, 3, 4, 6, 9, \dots \}$$

but ~~so~~ it is not a group since closure property not satisfied.

Union of 2 subgroups need not be a subgroup

Permutation Group

A bijective mapping of a non empty set $S \rightarrow S$ is called a permutation of S .

ex: If $S = \{a, b\}$ then the 2 possible permutations of $\{a, b\}$ are $\{\{a, b\} \& \{b, a\}\}$

We will represent this permutation as

$$P_1 = \begin{pmatrix} a & b \\ a & b \end{pmatrix} \quad P_2 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

Now the set $\{P_1, P_2\}$ is the set of all possible permutations of the elements of S .

We can define a binary operation on these set which is known as right composition of permutations.

$$\text{here } P_1 * P_2 = \begin{pmatrix} a & b \\ a & b \end{pmatrix} * \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

$$= \underline{\underline{P_2}}$$

$$P_2 * P_1 = \begin{pmatrix} a & b \\ b & a \end{pmatrix} * \begin{pmatrix} a & b \\ a & b \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \underline{\underline{P_2}}$$

first set
 $a \rightarrow a$ and in
 second
 $a \rightarrow b$
 $\therefore a \rightarrow b$

Permutation Group

The set G of all permutations on a nonempty set S under binary operation $*$ of right composition of permutations is a group

$\{G, *\}$ called the permutation group.

If $S = \{1, 2, 3, \dots, n\}$ the permutation group is also called the symmetric group of degree n and denoted by S_n .

The no. of elements of $S_n = n!$ since there are $n!$ permutations of n element

Symmetric group of 3 elements

Consider the set $\{S_3, *\}$

Where $S = \{1, 2, 3\}$ S.T $\{S, *\}$ is a group under the operation of right composition of permutations.

Ans. $S = \{1, 2, 3\}$

There will be $3! = 6$ permutations of the elements $1, 2, 3$ of S .

$$\text{ie, } S_3 = \{P_1, P_2, P_3, P_4, P_5, P_6\}$$

Where

$$P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad P_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

The composition table of permutations on $\{3\}$ is given below

$*$	P_1	P_2	P_3	P_4	P_5	P_6
P_1	P_1	P_2	P_3	P_4	P_5	P_6
P_2	P_2	P_1	P_4	P_3	P_6	P_5
P_3	P_3	P_6	P_5	P_2	P_1	P_4
P_4	P_4	P_5	P_6	P_1	P_2	P_3
P_5	P_5	P_4	P_1	P_6	P_3	P_2
P_6	P_6	P_3	P_2	P_5	P_4	P_1

From the table all the axioms of a group are easily verified.

$*$ is associative for exa:

$$(P_2 * P_4) * P_6 = P_3 * P_6 = P_4$$

$$P_2 * (P_4 * P_6) = P_2 * P_3 = P_4$$

Similarly P_1 acts as the identity element

$$P_i * P_i = P_i * P_1 = P_i \text{ for } i=1, 2, 3, \dots, 6$$

Each element possesses inverse also

$$P_1^{-1} = P_1, \quad P_2^{-1} = P_1, \quad P_3^{-1} = P_5, \quad P_4^{-1} = P_4, \quad P_5^{-1} = P_3$$

$$P_6^{-1} = P_6$$

Hence $(\{ \}, *)$ is a group.

But it is not abelian because

$$P_2 * P_3 = P_4 \quad \text{but}$$

$$P_3 * P_2 = P_6$$

$\therefore *$ is not commutative

Note → Symmetric group $(\{ \}, *)$ is not
abelian